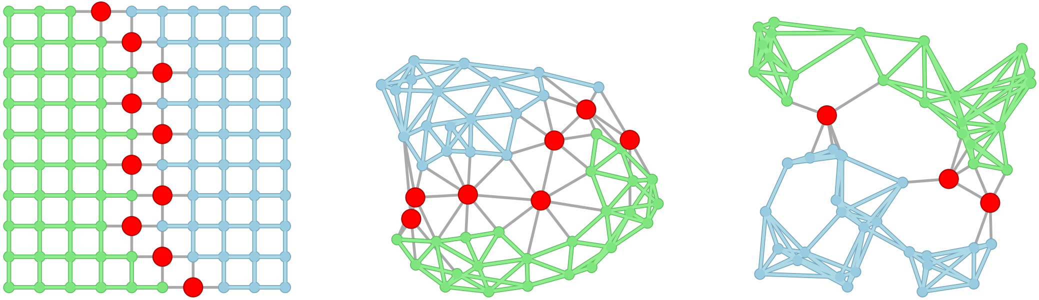
Reweighted Spectral Partitioning Works

A Simple Algorithm for Vertex Separators in Special Graph Classes

Jack Spalding-Jamieson

Balanced Vertex Separators



Vertex Separator: Small set of vertices whose removal disconnects into small components.

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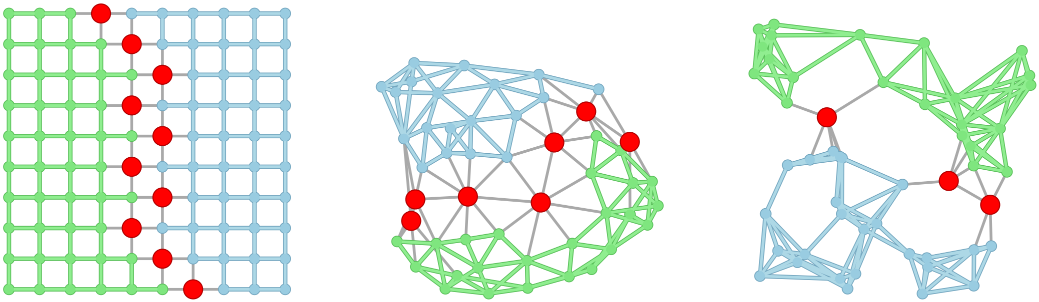

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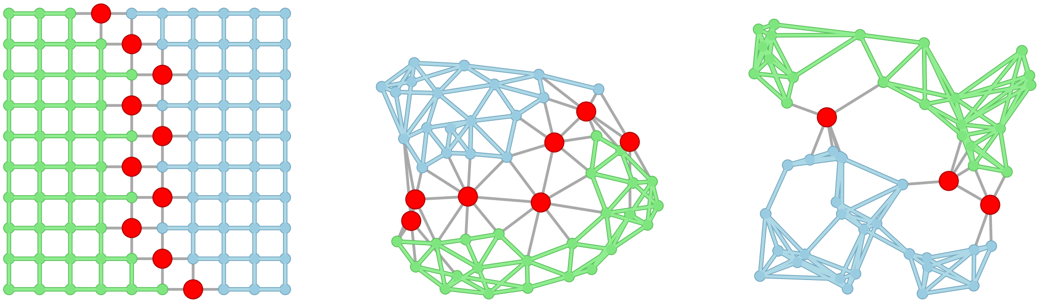

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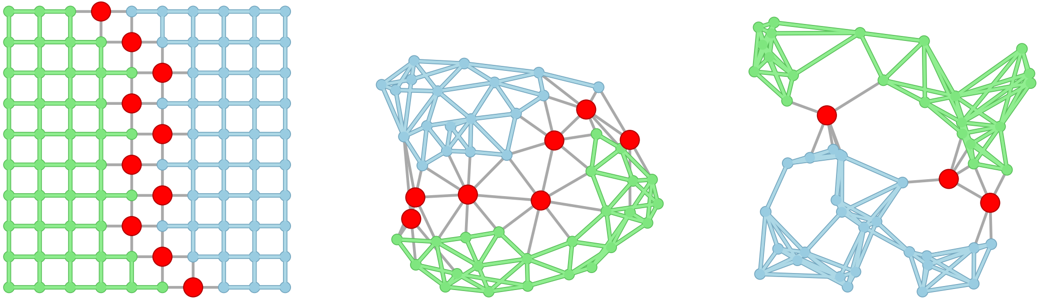

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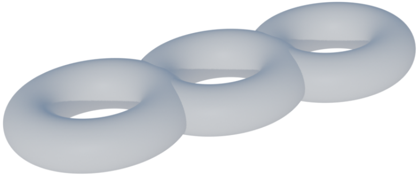
Small separator = many fast algorithms!

Other Separator Theorems (1)

Genus-g graph: Embeddable on genus-g surface without crossings.



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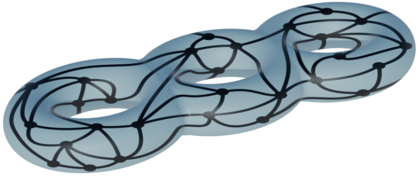

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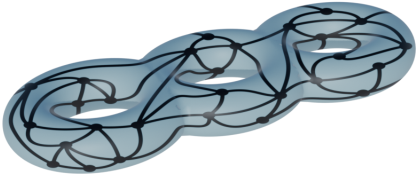

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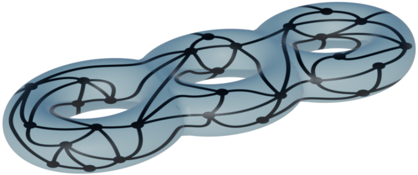

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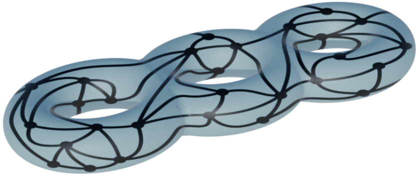

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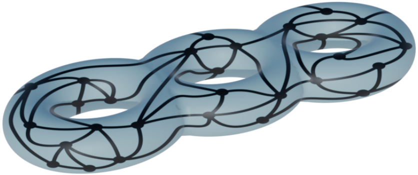

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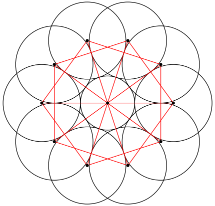
K_h-minor-free graph: Separator size $\mathcal{O}\left(h\sqrt{n}\right)$. Can be found in $\mathcal{O}(n^2)$ time, **provided that h is constant**.

Other Separator Theorems (2)

k-ply d-dimensional sphere-intersection graph



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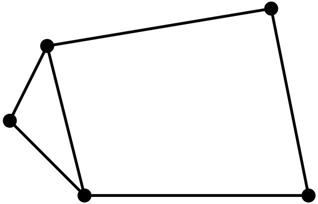

Theorem (MTTV97)

Separator size $\mathcal{O}\left(dk^{\frac{1}{d}}n^{1-\frac{1}{d}}\right)$.

Can be found in $\mathcal{O}(f(d) + nd^2)$ *time, for a function* f, ***if the points are provided***.

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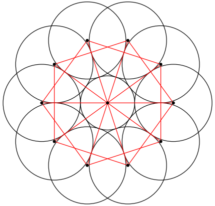

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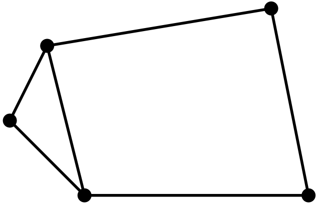

Theorem (New, Side-Result)

Separator size $\mathcal{O}\left(\sqrt{\min\{d, \log\Delta\}}k^{\frac{1}{d}}n^{1-\frac{1}{d}}\right)$.

Can be found in polynomial time, ***if the points are provided****.*

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In practice: Given graph, don't know class. Want guarantees if class is nice.

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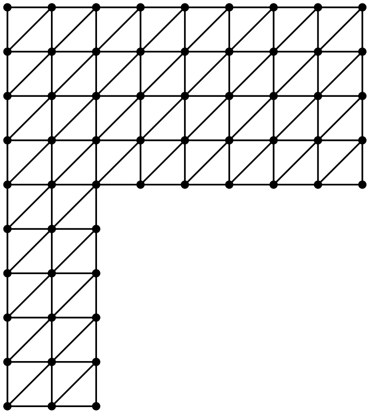
The algorithm we consider: **Reweighted Spectral Partitioning**.

Results: Poly-time Separator Sizes

Graph class	This work	Previous work
Genus-g	$\mathcal{O}(\min\{(\log g)^2\sqrt{gn}, \boldsymbol{\log \Delta\sqrt{gn}}\})$	$\mathcal{O}(\min\{\boldsymbol{(\log g)\sqrt{gn}}, \mathrm{poly}(\Delta)\sqrt{gn}\})$
K_h-minor-free	$\mathcal{O}(\min\{\log h, \sqrt{\log \Delta}\}(h \log h \log\log h)\sqrt{n})$	$\boldsymbol{\mathcal{O}((\log h)h\sqrt{n})}$
k-ply ball-intersection in $\mathbb{R}^d$	$\boldsymbol{\mathcal{O}(\sqrt{\log \Delta} \cdot k^{1/d} n^{1-1/d})}$	$\mathcal{O}(\sqrt{\log n} \cdot dk^{1/d} n^{1-1/d})$
k-nearest-neighbour in $\mathbb{R}^d$	$\boldsymbol{\mathcal{O}(\sqrt{\log \Delta} \cdot k^{1/d} n^{1-1/d})}$	$\mathcal{O}(\sqrt{\log n} \cdot dk^{1/d} n^{1-1/d})$

Reweighted spectral partitioning separator size guarantees (via this work) vs. previous algorithms.

Separators, Expansion, and Cuts



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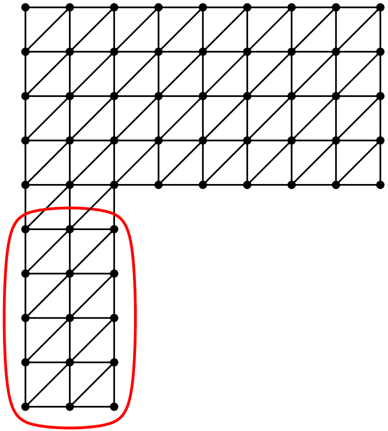

Want small boundary to area ratio

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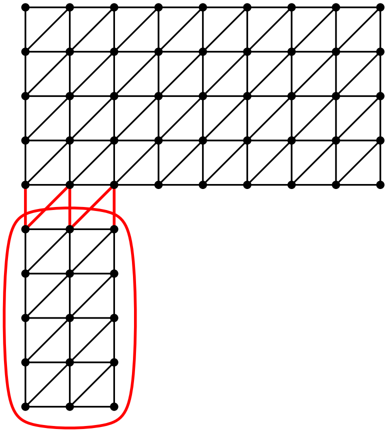

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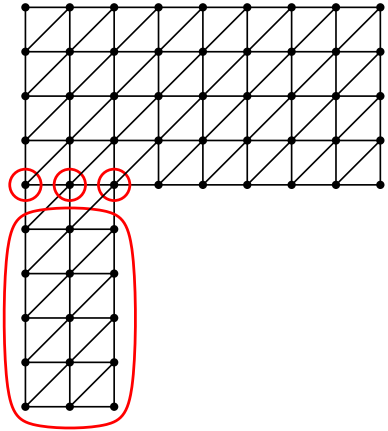

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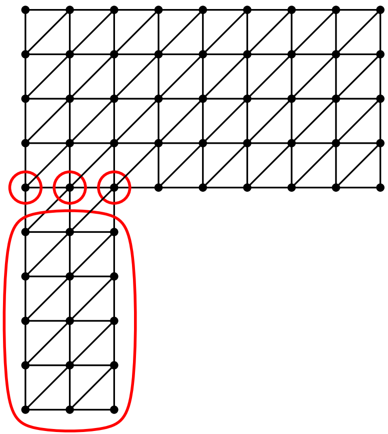

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Known algorithm:

For induced subgraph $H \subset G$, find cut S with $\psi(S) \leq \frac{\alpha}{|H|^{\varepsilon}}$

$\implies$ can get balanced vertex separator of size $O(\alpha n^{1-\varepsilon})$.

(Requires $|S| \leq \frac{n}{2}$)

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Similar results for many other classes (also in other works).

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$$\frac{\phi(G)^2}{2\Delta} \le \lambda_2(G) \le 2\phi(G)$$

Spectral partitioning algorithm: Compute $\lambda_2(G)$, obtain S with $\phi(S)$ bounded.

Vertex expansion $\psi(G) := \min_{|S| \le \frac{n}{2}} \psi(S)$: Small = fast algorithms.

Max Reweighted Spec Gap ($\gamma^{(n)}$): A poly-time computable quantity.

Theorem (Cheeger-Style Inequality [Roc05, OTZ22, JPV22, KLT22])

For a graph G with max degree Δ,

$$\frac{\psi(G)^2}{\log \Delta} \lesssim \gamma^{(n)}(G) \lesssim \psi(G).$$

Reweighted spectral partitioning algorithm: Compute $\gamma^{(n)}(G)$, obtain S with $\psi(S)$ bounded.

Refining Reweighted Spectral Partitioning

Theorem (Refined Cheeger-Style Inequality [New])

For a graph G with n vertices and maximum degree Δ,

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E.g. G planar $\implies \alpha(G) \in \mathcal{O}(1)$.

Reweighted Spectral Partitioning **Works**

Reweighted spectral partitioning works: Direct class-specific upper bounds for $\gamma^{(n)}(G)$.

Graph class	$\gamma^{(n)} \leq \gamma^{(1)} \lesssim$
Planar	$\frac{1}{n}$
Genus-g	$\frac{g \min\{(\log g)^2, \log \Delta\}}{n}$
K_h-minor-free	$\frac{(h \log h \log \log h)^2}{n}$

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E.g. G planar $\implies \gamma^{(n)}(G) \lesssim \frac{1}{n} \implies \psi(S) \lesssim \frac{1}{\sqrt{n}} \implies$ reproduces planar separator theorem!

Intuition for $\gamma^{(n)}(G)$

Definition

For a graph G, define:

$$
\begin{aligned}
\gamma^{(d)}(G) \quad := \quad & \min_{\substack{f:V\to\mathbb{R}^d \\ y:V\to\mathbb{R}_{\geq 0}}} \quad \frac{\sum_{v\in V} y(v)}{\sum_{x\in V} ||f(x)||_2^2} \\
& \text{subject to} \qquad \sum_{v\in V} f(v) \quad = \quad \bar{0} \\
& \qquad\qquad\qquad y(u) + y(v) \quad \geq \quad ||f(u) - f(v)||_2^2 \quad \forall uv \in E
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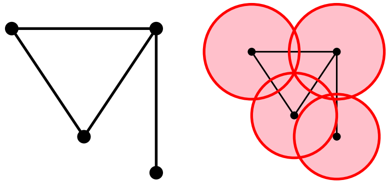
- For $v \in V$: Create a ball in $\mathbb{R}^d$ centred at $f(v)$, radius $\approx \sqrt{y(v)}$.



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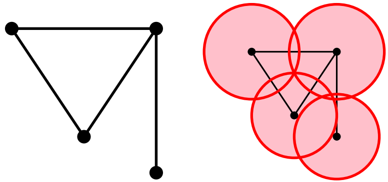

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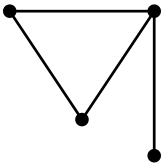

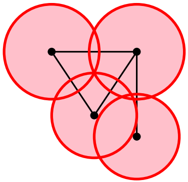

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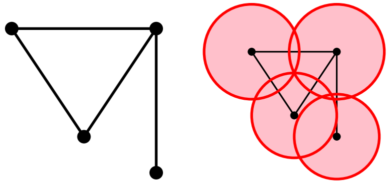
- Constraint: Adjacent balls *must* intersect.



Expanding the Cheeger-Style Inequality

Theorem (Refined Cheeger-Style Inequality, expanded)

For a graph G with n vertices and maximum degree Δ,

$$\frac{\psi(G)^2}{\min\{\log\Delta, [\alpha(G)]^2\}} \lesssim \frac{\gamma^{(1)}(G)}{\min\{\log\Delta, [\alpha(G)]^2\}} \lesssim \gamma^{(n)}(G) \lesssim \gamma^{(1)}(G) \lesssim \psi(G).$$

Reminder: $\gamma^{(n)}$ is the poly-time computable quantity (it is an SDP).

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Method: Don't change sphere radii, use random projection on centres.

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Ongoing follow-up work: This is now deterministic.

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Proof Step 2: Average 2-distortion bound $\implies$ normalizing denominator in objective only goes up by $\mathcal{O}(\alpha(G)^2)$.

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Reminder: $\gamma^{(n)}(G)$ small for a class of graphs $\implies$ small sparse cuts $\psi(S)$.

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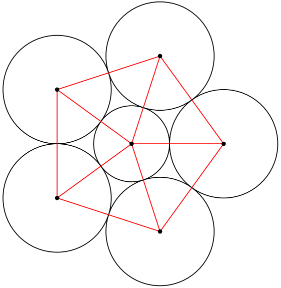
Geometric Bounds on $\gamma^{(n)}(G)$

Rich theory of **circle packings**!



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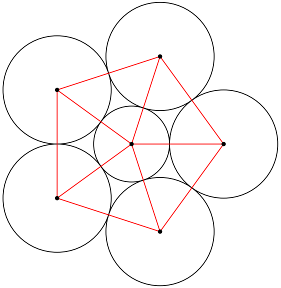

Combinatorial Bounds on $\gamma^{(n)}(G)$

Congestion bounds via crossing numbers!

E.g., crossing number lemma:

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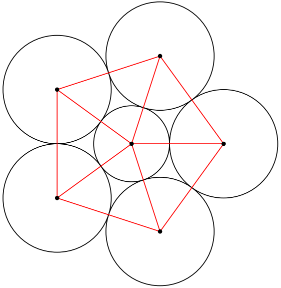

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Either kind $\implies$ $\gamma^{(1)}(G) \lesssim \frac{1}{n}$ for G planar.

Geometric Bounds: Planar Case

Construction from Spielman-Teng:

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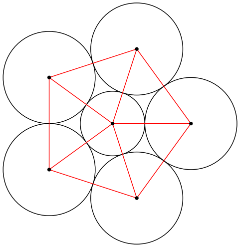
Step 1: Planar circle packing theorem. Every planar graph admits touching circles representation.



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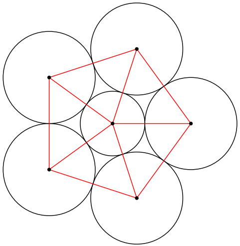

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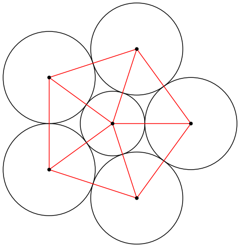

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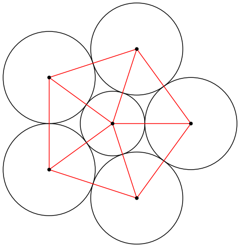

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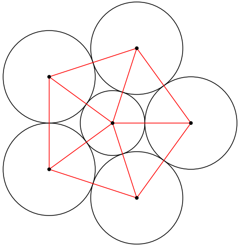

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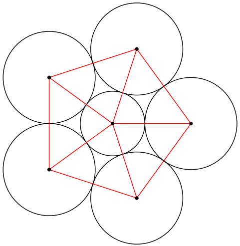

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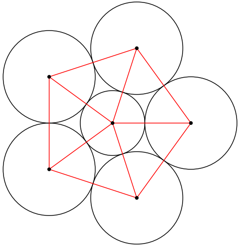

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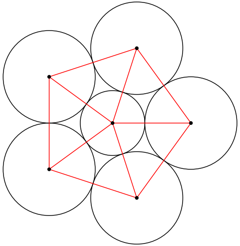

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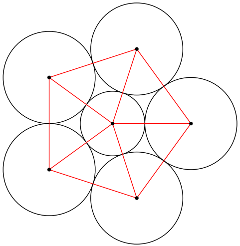

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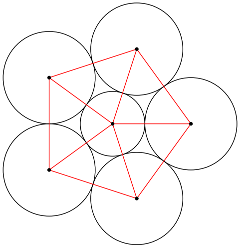

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Area bound: $\sum_{v \in V} y(v) \lesssim 1$.

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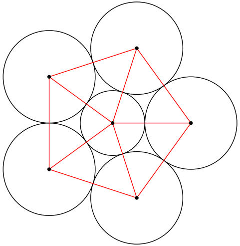
Construction from Spielman-Teng:

Step 1: Planar circle packing theorem. Every planar graph admits touching circles representation.



Step 2: Stereographic projection: Put circle packing onto unit sphere (centroid at origin).

Step 3: Total area of representation $\leq$ area of unit sphere (4π)

Bounding $\gamma^{(3)}$:

Use $y(v) := 2 \cdot \text{rad}(v)^2$,
$f(v) :=$ centre of disk on sphere.

Centroid at origin:
$\sum_{v \in V} f(v) = \bar{0}$

Touching circles ($uv \in E$):
$y(u) + y(v) \geq (\text{rad}(u) + \text{rad}(v))^2$
$\geq \|f(u) - f(v)\|_2^2$

Unit sphere: $\sum_{x \in V} \|f(x)\|_2^2 = n$

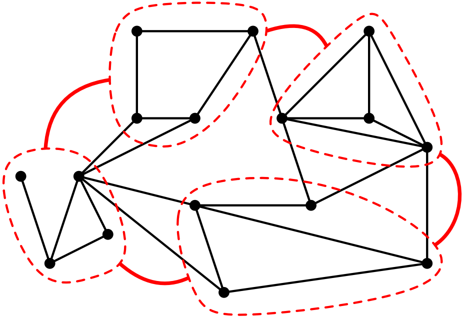
Area bound: $\sum_{v \in V} y(v) \lesssim 1$.

Result: $\dfrac{\sum_{v \in V} y(v)}{\sum_{x \in V} \|f(x)\|_2^2} \lesssim \dfrac{1}{n}$

Geometric Bounds: Uniform Shallow Minors (for Genus-g Graphs)

New structure: **Uniform shallow minors**.

- Start with graph G.
- Will form new graph H.



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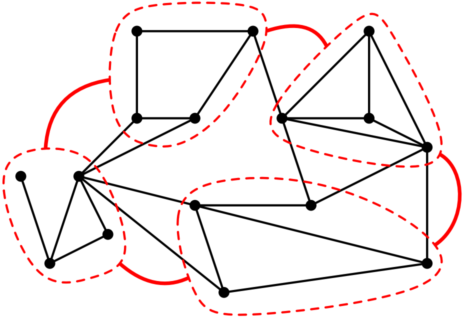

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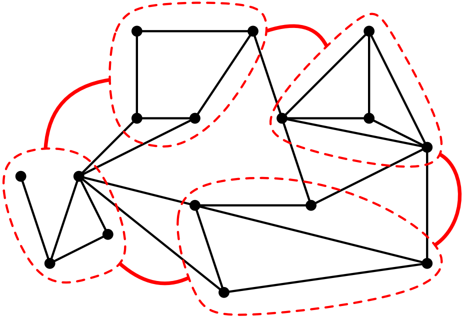

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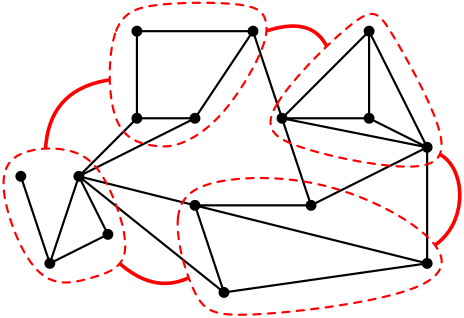

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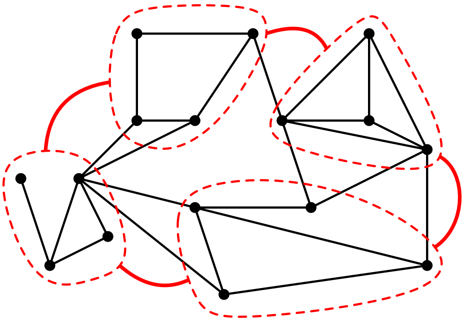

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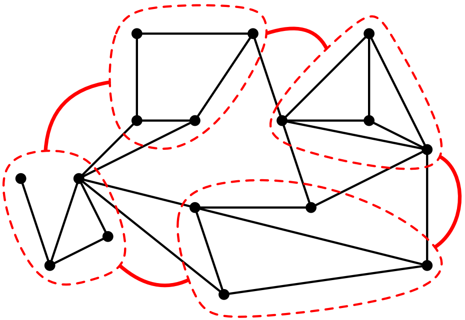

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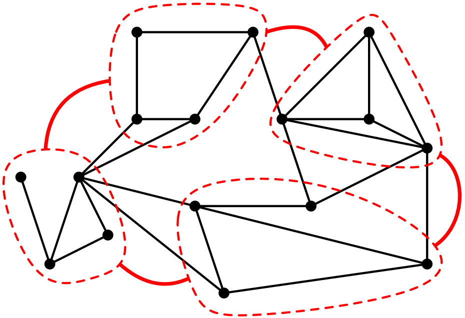

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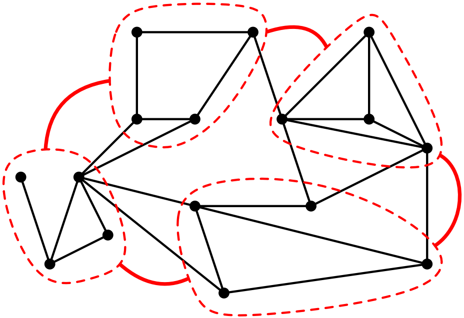

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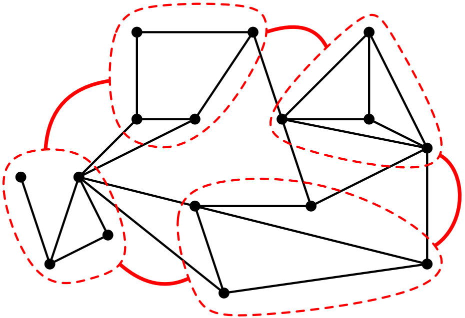

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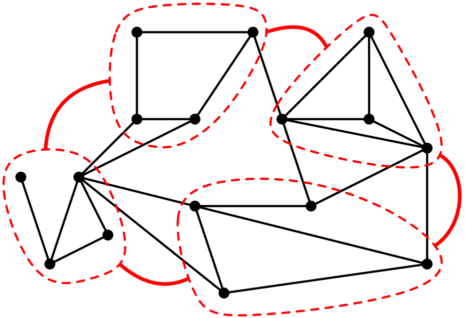

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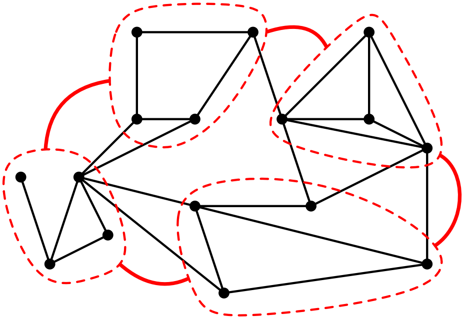

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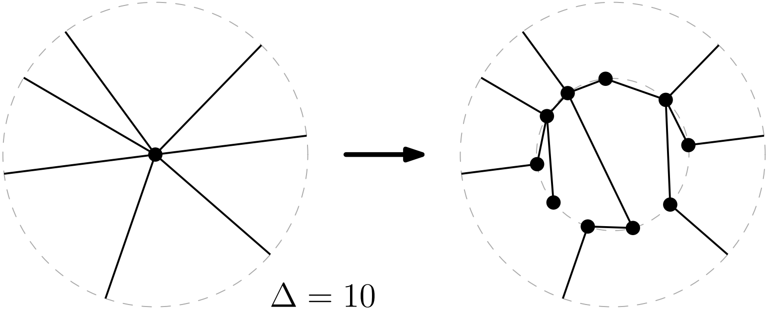
Four steps:

1. Reduce to constant-degree graph [**uniform shallow minors**].



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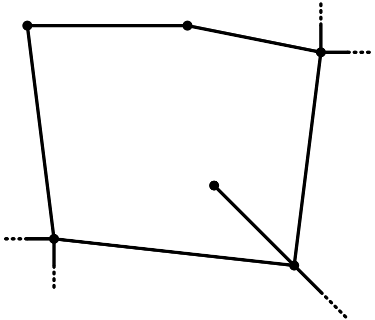

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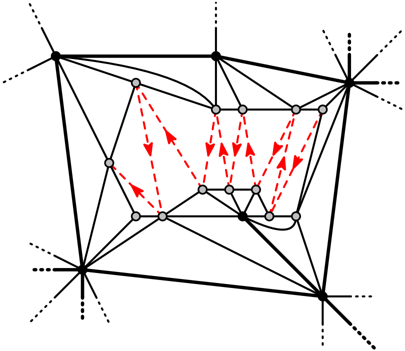

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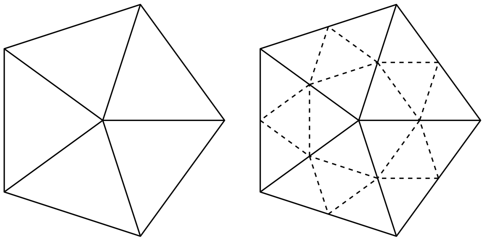

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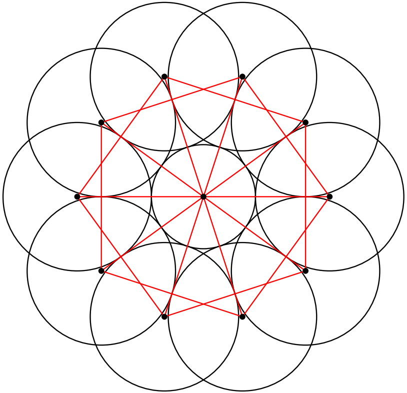

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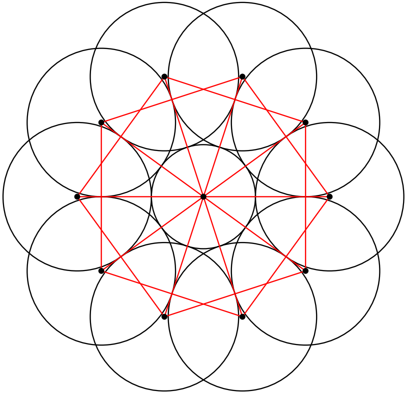

Result: $\gamma^{(1)}(G) \lesssim \frac{g \log \Delta}{n}$

Combinatorial Bounds: Overview for Genus-g Graphs

$$\mathrm{cr}_g(K_n) \gtrsim \frac{n^4}{g}$$

$\downarrow$

$$\mathrm{con}_2^*(G) \gtrsim \frac{n^2}{\sqrt{g}}$$

$\downarrow$ Integrality gap

$$\mathrm{con}_2(G) \gtrsim \frac{n^2}{\sqrt{g}}$$

$\updownarrow$ Strong duality

$$\bar{s}_2(G) \gtrsim \frac{n^2}{\sqrt{g}}$$

$\rightarrow$ Cauchy-Schwarz

$$\sup_{\substack{\omega : V \to \mathbb{R}_{\geq 0} \\ ||\omega||_2 \leq 1}} \sum_{u,v \in V} [d_G^{\omega}(u,v)]^2 \gtrsim \frac{n^2}{g}$$

$\downarrow$ Metric embedding (from $\bar{s}_2(G)$)

$$\overline{Q}_1^{(1)}(G) \gtrsim \frac{1}{\sqrt{g}\log g}$$

$\rightarrow$ Cauchy-Schwarz

$\downarrow$ Metric embedding (from the sup bound)

$$\overline{Q}_2^{(1)}(G) \gtrsim \frac{n^2}{g(\log g)^2}$$

$\leftrightarrow$

$$\gamma^{(1)}(G) \lesssim (\log g)^2 \frac{g}{n}$$

$\uparrow$ Relaxation

$$\gamma^{(n)}(G) \lesssim (\log g)^2 \frac{g}{n}$$

$\updownarrow$ Strong duality

$$\lambda_2^*(G) \lesssim (\log g)^2 \frac{g}{n}$$

Combinatorial Bounds: Overview for Genus-g Graphs

BLR10

- $\mathrm{cr}_g(K_n) \gtrsim \frac{n^4}{g}$
- ↓
- $\mathrm{con}_2^*(G) \gtrsim \frac{n^2}{\sqrt{g}}$
- ↓ Integrality gap
- $\mathrm{con}_2(G) \gtrsim \frac{n^2}{\sqrt{g}}$
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- $\bar{s}_2(G) \gtrsim \frac{n^2}{\sqrt{g}}$

$\bar{s}_2(G) \gtrsim \frac{n^2}{\sqrt{g}}$ → Cauchy-Schwarz → $\sup_{\substack{\omega: V \to \mathbb{R}_{\geq 0} \\ ||\omega||_2 \leq 1}} \sum_{u,v \in V} [d_G^\omega(u,v)]^2 \gtrsim \frac{n^2}{g}$

$\bar{s}_2(G) \gtrsim \frac{n^2}{\sqrt{g}}$ ↓ Metric embedding ↓ $\overline{Q}_1^{(1)}(G) \gtrsim \frac{1}{\sqrt{g}\log g}$

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BDX04
Roch05
OTZ22

- $\lambda_2^*(G) \lesssim (\log g)^2 \frac{g}{n}$
- ↕ Strong duality
- $\gamma^{(n)}(G) \lesssim (\log g)^2 \frac{g}{n}$
- ↑ Relaxation
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Fin

Reweighted Spectral Partitioning Works:
A Simple Algorithm for Vertex Separators in Special Graph Classes

Jack Spalding-Jamieson

https://arxiv.org/pdf/2506.01228

Lots more results in the paper!

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