Recent Advances in Metric Embeddings for Graph Partitioning

Jack Spalding-Jamieson

Papers

DR**S**W 2026	Revisiting separators in string graphs, region intersection graphs, and induced-minor-free graphs
K**S** 2026	Fast approximation algorithms for balanced edge and vertex separators
S 2025	Reweighted spectral partitioning works: a simple algorithm for vertex separators in special graph classes
S 2026	Optimal average distortion for non-expansive embeddings of Euclidean metrics into the line: with applications to cut-matching games and vertex expansion

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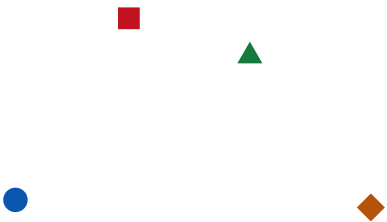
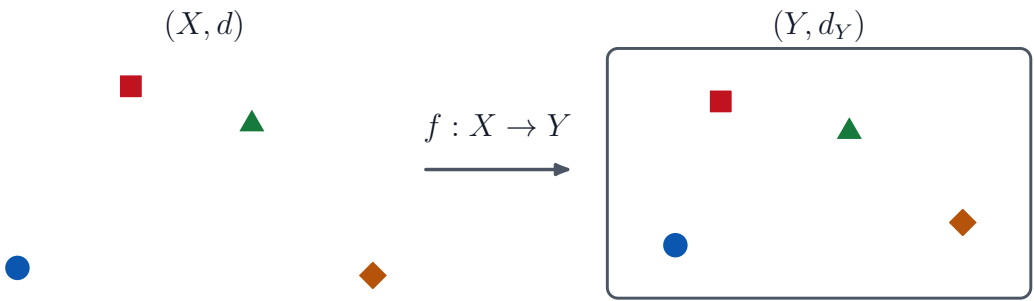
this talk: **metric embeddings**

Metric embeddings

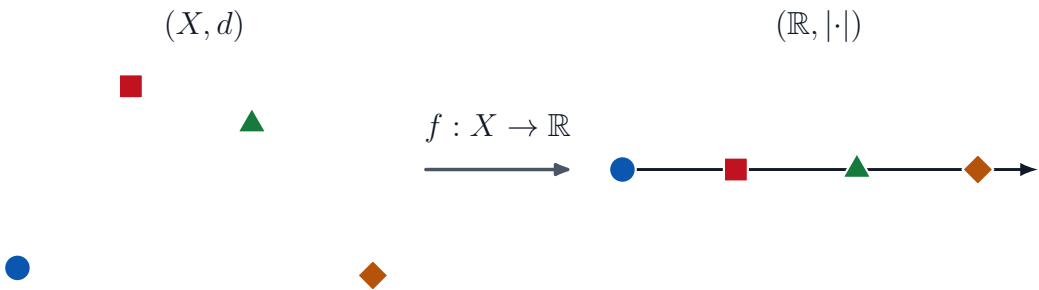
(X, d)



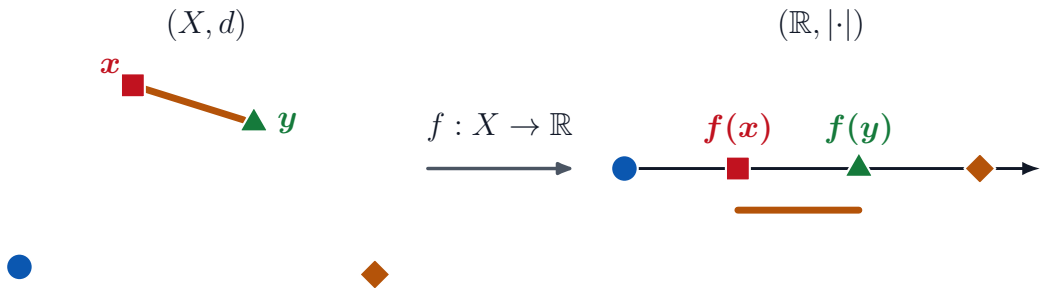
Metric embeddings



Metric embeddings into the line



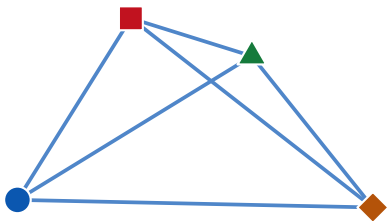
Non-expansive metric embeddings into the line



$|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in X$

Non-expansive metric embeddings into the line with bounded average distortion

(X, d)



$(\mathbb{R}, |\cdot|)$

$f : X \to \mathbb{R}$



$$\mathrm{avd}(f) := \frac{\sum_{x,y} d(x,y)}{\sum_{x,y} |f(x) - f(y)|}$$

Overview

warmup

K_h-minor-free

$O(\log h)$

vertex expansion

Overview

warmup

K_h-minor-free

$O(\log h)$

vertex expansion

1

$K_h^{(1)}$-induced-minor-free

$O(\log h)$

separators

Overview

warmup	1	2
K_h-minor-free	$K_h^{(1)}$-induced-minor-free	K_h-minor-free, SDP
$O(\log h)$	$O(\log h)$	$O(\sqrt{\log h})$
vertex expansion	separators	edge expansion

Overview

warmup

K_h-minor-free

$O(\log h)$

vertex expansion

1

$K_h^{(1)}$-induced-minor-free

$O(\log h)$

separators

2

K_h-minor-free, SDP

$O(\sqrt{\log h})$

edge expansion

3

Euclidean

$\Theta(\sqrt{\frac{\log n}{\log \log n}})$

spectral vertex expansion

Overview

warmup	1	2	3	
K_h-minor-free	$K_h^{(1)}$-induced-minor-free	K_h-minor-free, SDP	Euclidean	
$O(\log h)$	$O(\log h)$	$O(\sqrt{\log h})$	$\Theta(\sqrt{\frac{\log n}{\log \log n}})$	???
vertex expansion	separators	edge expansion	spectral vertex expansion	

Warmup: vertex expansion

1. Induced-minor-free graphs
2. Two metrics at once
3. Euclidean metrics

???

Warmup: vertex expansion

1. Induced-minor-free graphs
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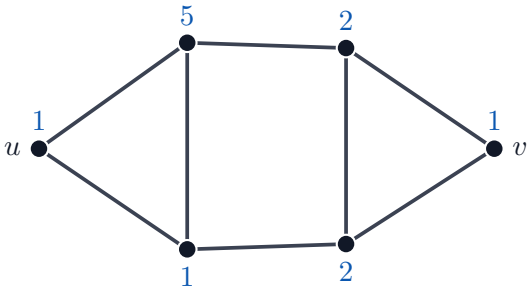
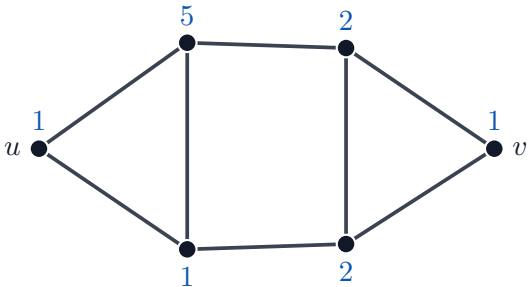
the warmup: known results, not new work

Vertex-weighted shortest paths



weights w_x

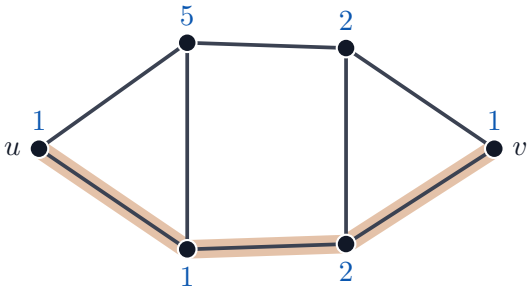
Vertex-weighted shortest paths



weights w_x

edge xy has length $w_x + w_y$

Vertex-weighted shortest paths

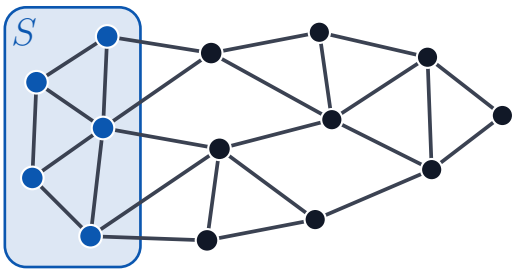


weights w_x

edge xy has length $w_x + w_y$

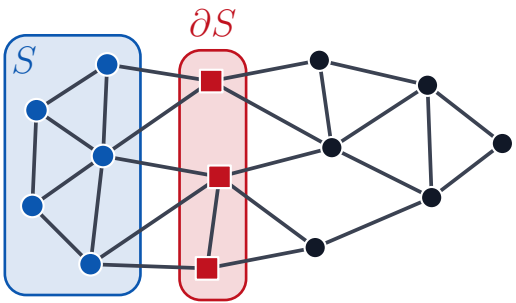
$d_w(u, v) = 2 + 3 + 3 = 8$

Vertex expansion



$S \subseteq V,\ |S| \leq n/2$

Vertex expansion



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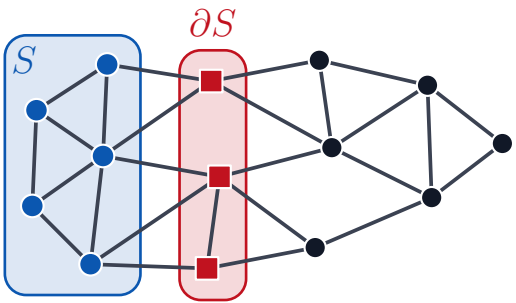
∂S: neighbours of S outside S

Vertex expansion



$S \subseteq V,\ |S| \leq n/2$

∂S: neighbours of S outside S

$$\psi(G) := \min_S \frac{|\partial S|}{|S|}$$

LP relaxation

relaxation

minimize $\quad n \cdot \dfrac{\sum_v w_v}{\sum_{u,v} d_w(u,v)}$

subject to $\quad w \geq 0$

$$\lesssim \psi(G)$$

LP relaxation

relaxation

minimize $n \cdot \dfrac{\sum_v w_v}{\sum_{u,v} d_w(u,v)}$

subject to $w \geq 0$

$\lesssim \psi(G)$

line-embedded

minimize $n \cdot \dfrac{\sum_v w_v}{\sum_{u,v} |f(u) - f(v)|}$

subject to $|f(u) - f(v)| \leq w_u + w_v$

for every edge uv

$\approx \psi(G)$

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want: line-embedded $\leq O(\alpha) \cdot$ relaxation ($\Rightarrow O(\alpha)$-approximation)

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$\approx \psi(G)$

want: line-embedded $\leq O(\alpha) \cdot$ relaxation ($\Rightarrow O(\alpha)$-approximation)

suffices: f non-expansive for d_w (= the edge constraints), $\operatorname{avd}(f) \in O(\alpha)$

Any metric

> ***Theorem*** *via Bourgain 1985*
>
> Every n-point metric has a non-expansive $f : X \to \mathbb{R}$ with $\operatorname{avd}(f) \in O(\log n)$.

Bourgain's embedding into L_1 $\rightarrow$ one of its coordinates $d(\cdot, A)$

Any metric

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$\implies$ vertex expansion: $O(\log n)$-approximation

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Bourgain's embedding into L_1 $\to$ one of its coordinates $d(\cdot, A)$

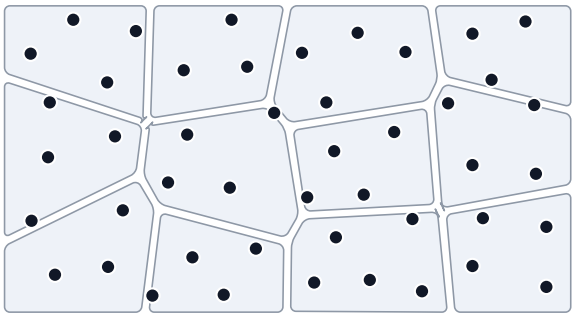
$\Longrightarrow$ vertex expansion: $O(\log n)$-approximation

tight on expanders

Line embeddings

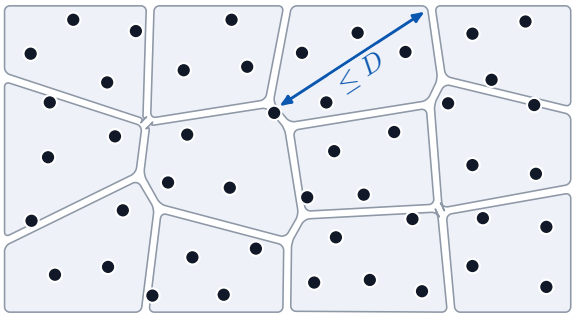
metric	avd	lower bound
any n-point metric	$O(\log n)$ Bourgain 1985	$\Omega(\log n)$ expanders

Padded decompositions



random partition

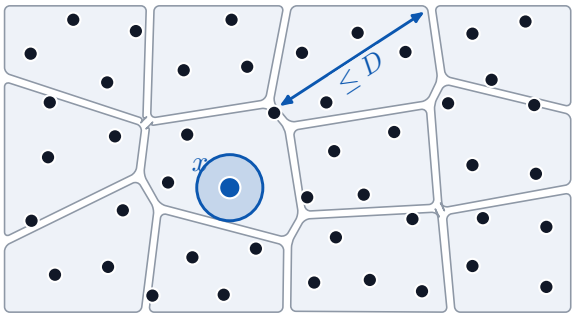
Padded decompositions



random partition

parts of diameter $\leq D$

Padded decompositions



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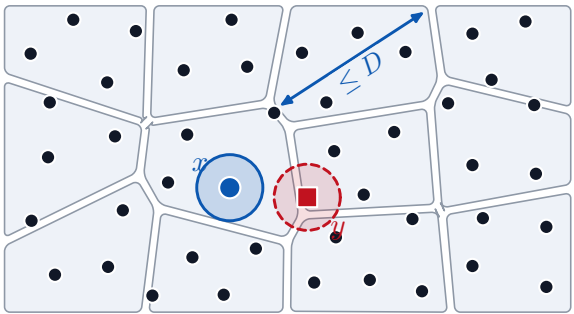
x padded: $B(x, D/\beta) \subseteq$ part of x

Padded decompositions



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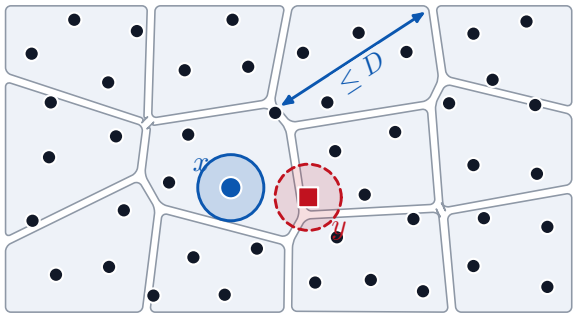
Padded decompositions



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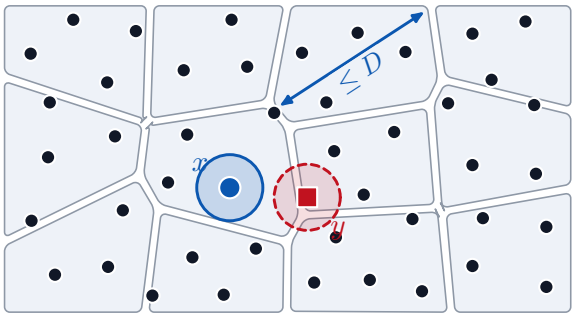
$\Pr[x \text{ padded}] = \Omega(1) \quad \forall x$

β: padding parameter

Padded decompositions



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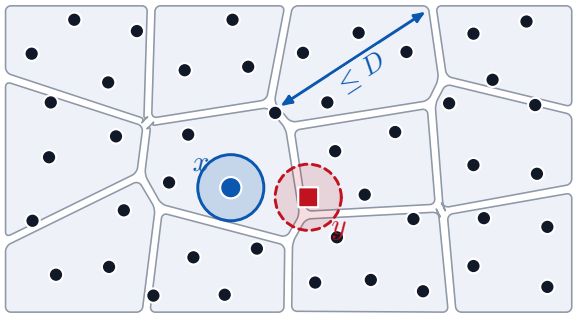
n-point metrics $\qquad \beta = O(\log n) \qquad$ Bartal 1996

Padded decompositions



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β: padding parameter

n-point metrics	$\beta = O(\log n)$	Bartal 1996
K_h-minor-free graphs	$\beta = O(\log h)$	CF 2025

Clustered or well-spread

$$\Delta := \frac{1}{n^2} \sum_{x,y} d(x, y)$$

Clustered or well-spread

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Clustered

a ball B of radius $\Delta/4$ has $\Omega(n)$ points



$$f(x) := d(x, B)$$
$$\mathrm{avd}(f) = O(1)$$

Clustered or well-spread

$$\Delta := \frac{1}{n^2} \sum_{x,y} d(x,y)$$

Clustered

a ball B of radius $\Delta/4$ has $\Omega(n)$ points



$f(x) := d(x, B)$

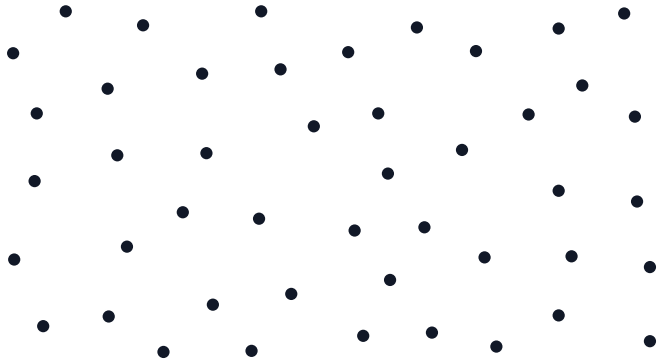
$\mathrm{avd}(f) = O(1)$

Well-spread

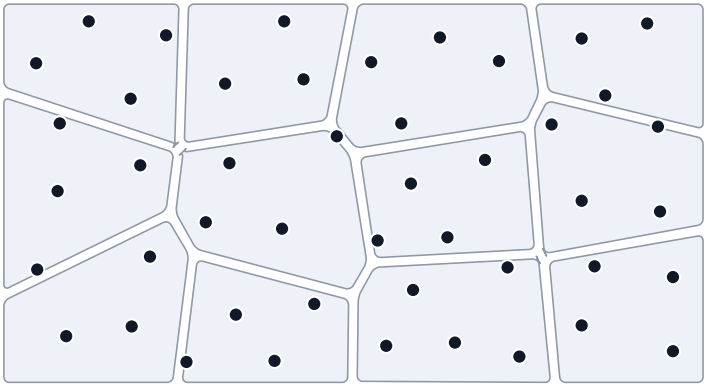
every ball of radius $\Delta/4$ has few points



Well-spread: padded decompositions

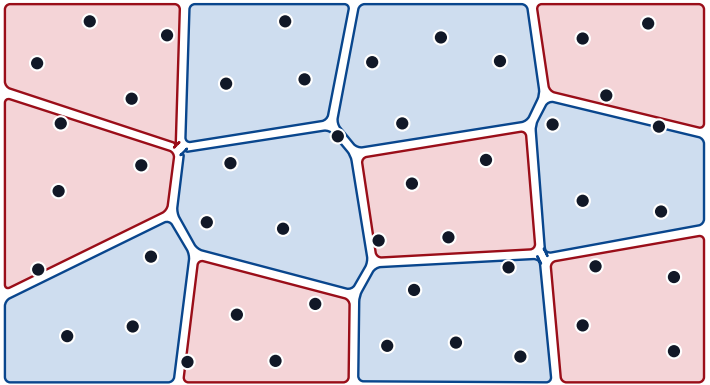


Well-spread: padded decompositions



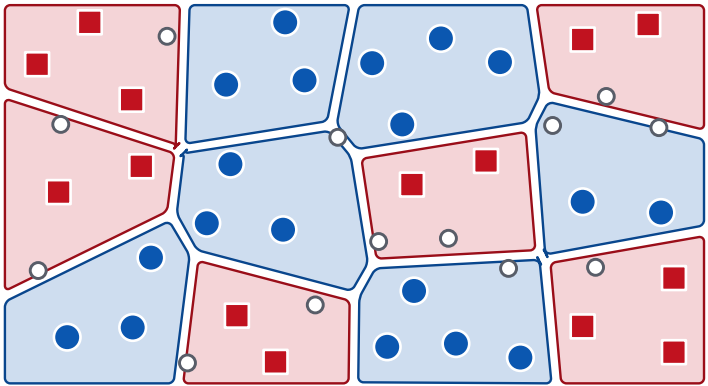
a padded decomposition, $D = \Delta/4$

Well-spread: padded decompositions



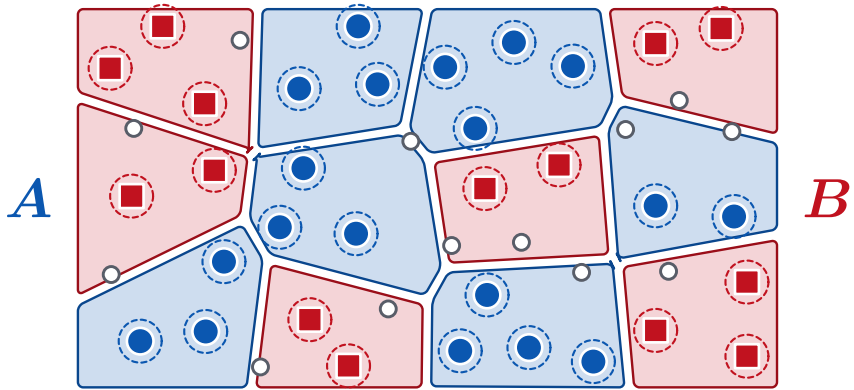
random colours

Well-spread: padded decompositions



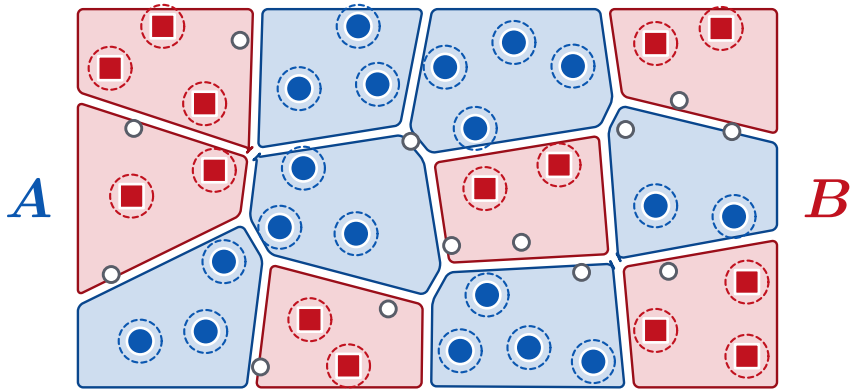
padded points

Well-spread: padded decompositions



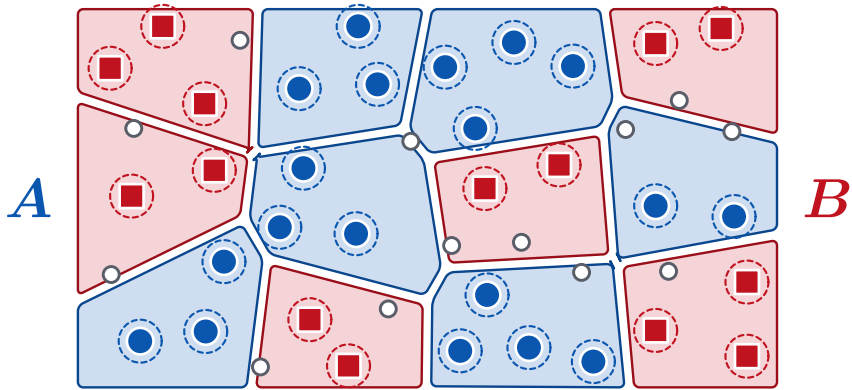
$$d(A, B) \geq D/\beta$$

Well-spread: padded decompositions



$$f(x) := d(x, A)$$

Well-spread: padded decompositions



$$f(x) := d(x, A) \qquad \mathrm{avd}(f) \in O(\beta)$$

Padded decompositions

Theorem *Rabinovich 2003; BLR 2010*

If (X, d) has padding parameter β, then some non-expansive $f : X \to \mathbb{R}$ has $\mathrm{avd}(f) \in O(\beta)$.

Padded decompositions

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K_h-minor-free: $\beta = O(\log h)$ CF 2025

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$\Longrightarrow$ vertex expansion: $O(\log h)$-approximation on K_h-minor-free graphs

Line embeddings

metric	avd	lower bound
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Line embeddings

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Warmup: vertex expansion

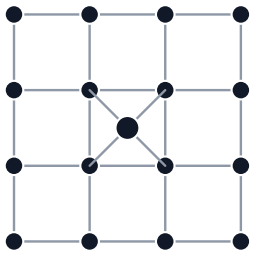
1. Induced-minor-free graphs

2. Two metrics at once

3. Euclidean metrics

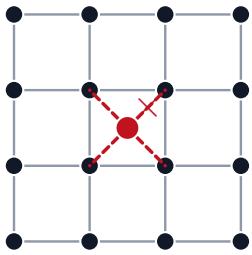
???

Induced minors



G

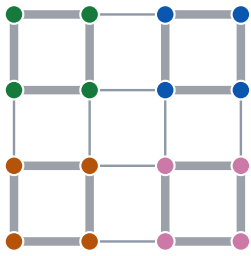
Induced minors



G

delete vertices

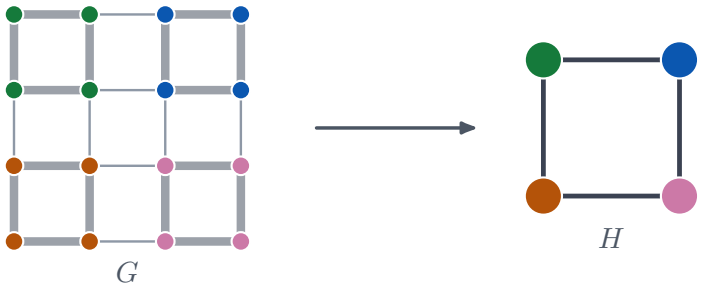
Induced minors



G

delete vertices contract edges

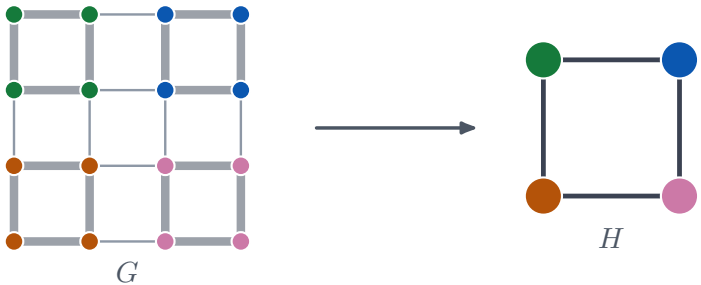
Induced minors



delete vertices contract edges

H: an induced minor of G

Induced minors

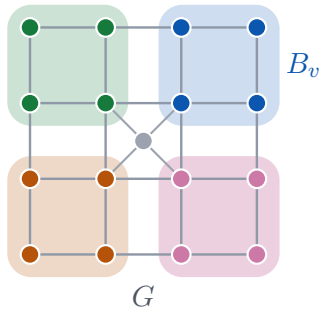


delete vertices contract edges

H: an induced minor of G

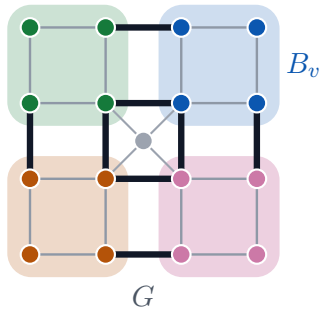
no edge deletions

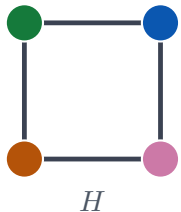
Induced minors



branch sets B_v: disjoint, connected

Induced minors

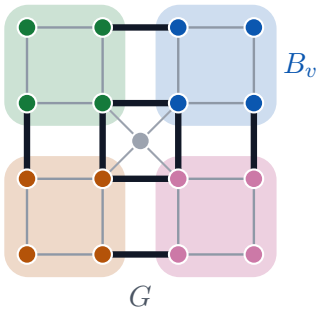


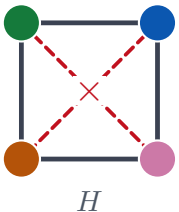


branch sets B_v: disjoint, connected

$uv \in E(H) \iff$ an edge between B_u and B_v

Induced minors

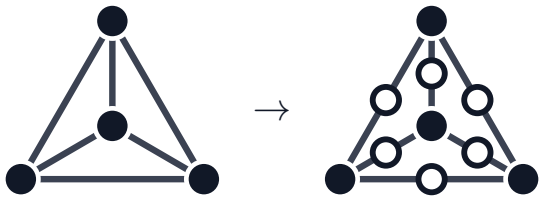




branch sets B_v: disjoint, connected

$uv \in E(H) \iff$ an edge between B_u and B_v

$K_h^{(1)}$



$K_h^{(1)}$: subdivide every edge of K_h

LP relaxation

relaxation

minimize $\quad n \cdot \dfrac{\sum_v w_v}{\sum_{u,v} d_w(u,v)}$

subject to $\quad w \geq 0$

$$\lesssim \psi(G)$$

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line-embedded

minimize $n \cdot \dfrac{\sum_v w_v}{\sum_{u,v} |f(u) - f(v)|}$

subject to $|f(u) - f(v)| \leq w_u + w_v$

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suffices: f non-expansive for d_w, $\operatorname{avd}(f) \in O(\log h)$

Padded decompositions?

$$\text{padding } \beta = O(\log h) \implies \operatorname{avd}(f) \in O(\log h)$$

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Open problem

If G is $K_h^{(1)}$-induced-minor-free, does every vertex-weighted shortest-path metric of G have padding parameter $O(\log h)$?

open even for padding $f(h)$

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instead: a direct embedding

Clustered or well-spread

$$\Delta := \frac{1}{n^2} \sum_{x,y} d(x, y)$$

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Clustered

a ball B of radius $\Delta/4$ has $\Omega(n)$ points



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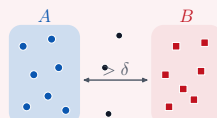

large A, B far apart (next)

$f(x) := d(x, A)$

Structure theorem

> ***Theorem*** *DRSW 2026*
> In a shortest-path metric of G, one of:

Structure theorem

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In a shortest-path metric of G, one of:

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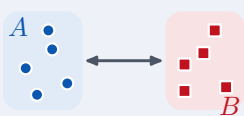

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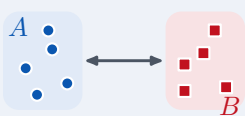
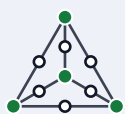


(3) $K_h^{(1)}$ induced minor



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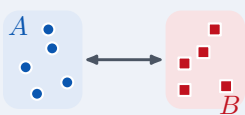

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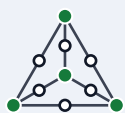

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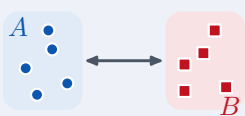

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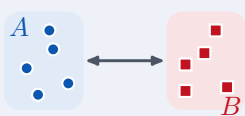

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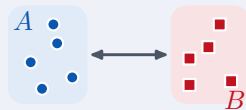

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impossible

Corollary: $K_h^{(1)}$-induced-minor-free $\implies$ non-expansive f with $\mathrm{avd}(f) \in O(\log h)$

Fat models

> ***Lemma*** *DRSW 2026*
> In any metric, one of:

Fat models

Lemma *DRSW 2026*

In any metric, one of:

(1) a ball of radius 4 with $> \frac{n}{4h^2}$ points



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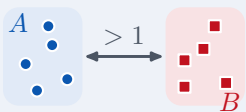

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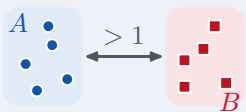
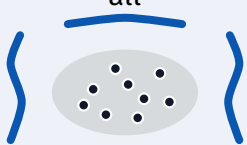


(3) 1-connected $A_1, \ldots, A_h$, > 4 apart; $\geq \frac{n}{3}$ points within 5 of all



1-connected: joined by steps of length ≤ 1

Line embeddings

metric	avd	lower bound
any n-point metric	$O(\log n)$ Bourgain 1985	$\Omega(\log n)$ expanders
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Warmup: vertex expansion

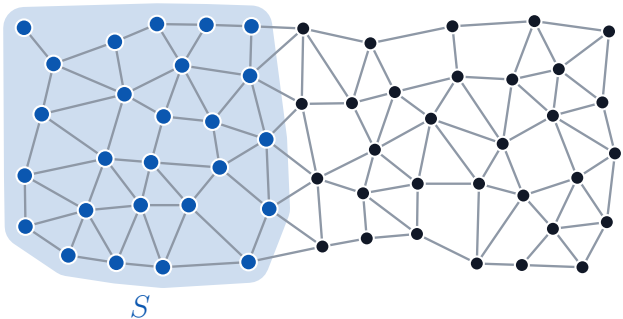
1. Induced-minor-free graphs

2. Two metrics at once

3. Euclidean metrics

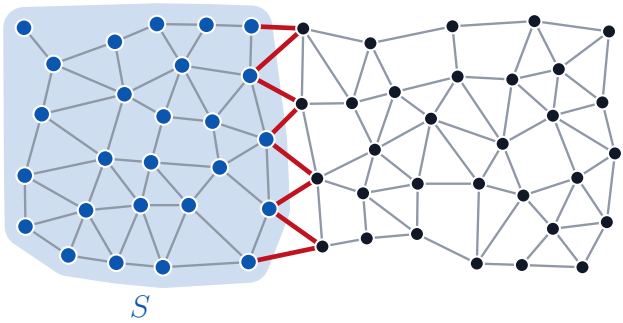
???

Edge expansion



$S \subseteq V,\ |S| \leq n/2$

Edge expansion



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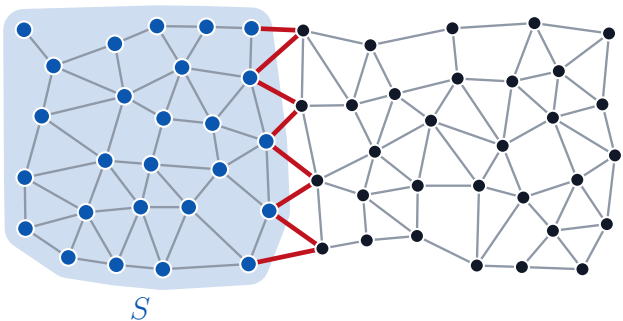

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$$\phi(G) := \min_S \frac{|E(S, \bar{S})|}{|S|}$$

L_2^2 metrics

d is an L_2^2 metric if $d(u, v) = \|x_u - x_v\|^2$ for points $x_v \in \mathbb{R}^k$, and d is a metric

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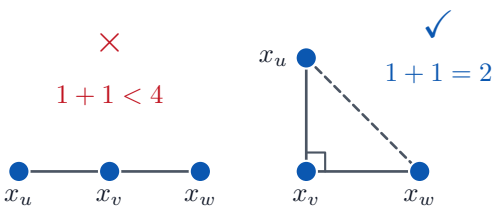

triangle inequality $\Leftrightarrow$ no obtuse angles

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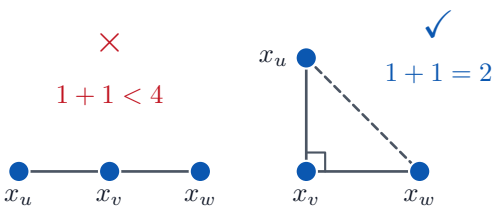

triangle inequality ⇔ no obtuse angles

also called: negative type

SDP relaxation

relaxation

minimize	$n \cdot \frac{\sum_{uv \in E} \|x_u - x_v\|^2}{\sum_{u,v} \|x_u - x_v\|^2}$
subject to	$\|x_u - x_v\|^2$ an L_2^2 metric

$$\lesssim \phi(G)$$

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line-embedded

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suffices: **(i)** $|f(u) - f(v)| \leq \|x_u - x_v\|^2$ on edges

(ii) $\mathrm{avd}(f) := \frac{\sum \|x_u - x_v\|^2}{\sum |f(u) - f(v)|} \in O(\alpha)$

Structure theorem

> ***Theorem*** *ARV 2004*
> In an L_2^2 metric, one of:

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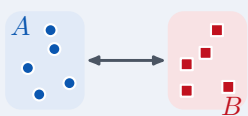

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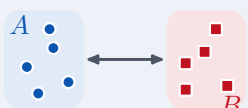

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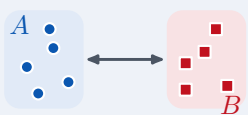

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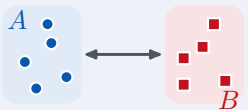

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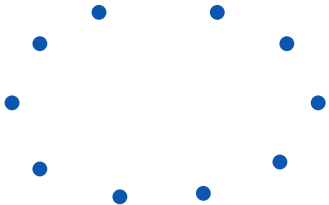

points x_v

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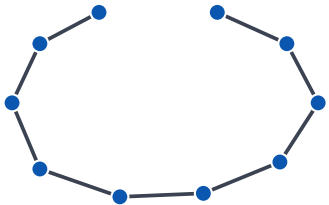

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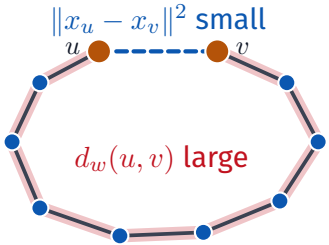

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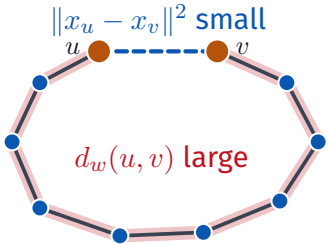

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non-expansive for $d_w \Leftrightarrow$ (i)

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e.g. K_h-minor-free: $O(\sqrt{\log h})$?

idea 1: embed $\|\cdot\|^2$

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both at once

non-expansive w.r.t. d_w (idea 2)

spread w.r.t. $\|\cdot\|^2$, from a projection (idea 1)

idea: **S** 2025

Two metrics at once

Theorem *KS 2026*

If every shortest-path metric of G has a non-expansive line embedding with $\mathrm{avd} \le \alpha$, then for the points x_v some f non-expansive for d_w has

$$\sum_{u,v} \|x_u - x_v\|^2 \;\le\; O(\sqrt{\alpha}) \cdot \sum_{u,v} |f(u) - f(v)|.$$

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K_h-minor-free: $O(\sqrt{\log h})$-approximation

Clustered or well-spread

$$\Delta := \frac{1}{n^2} \sum_{x,y} \|x - y\|^2$$

Clustered or well-spread

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Clustered

a ball B of radius $\Delta/4$ has $\Omega(n)$ points



$$f(x) := d_w(x, B)$$
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$$f(x) := d_w(x, B)$$
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Well-spread

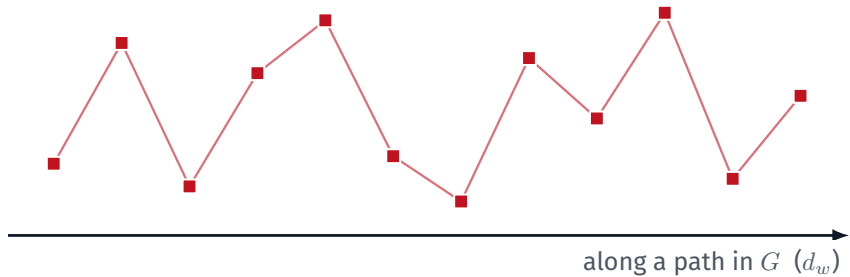
every ball of radius $\Delta/4$ has few points



Well-spread case

Algorithm

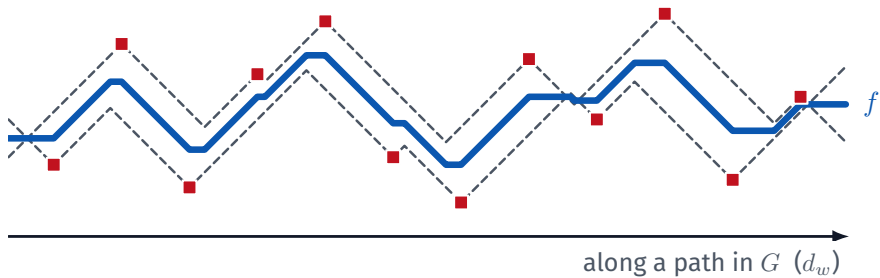
1. $g(v) := c\,\langle \gamma, x_v \rangle$ random direction γ, small scale c



Well-spread case

Algorithm

1. $g(v) := c\,\langle \gamma, x_v \rangle$ random direction γ, small scale c
2. $f(v) := \frac{1}{2}\Big(\max_u \{g(u) - d_w(u,v)\} + \min_u \{g(u) + d_w(u,v)\} \Big)$



Line embeddings

metric	avd	lower bound
any n-point metric	$O(\log n)$ Bourgain 1985	$\Omega(\log n)$ expanders
K_h-minor-free	$O(\log h)$ CF 2025	$\Omega(\log h)$ expanders
$K_h^{(1)}$-induced-minor-free	$O(\log h)$ DR**S**W 2026	$\Omega(\log h)$ expanders

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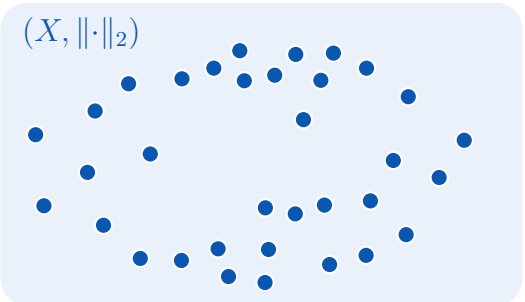
L_2^2, K_h-minor-free	$O(\sqrt{\log h})$ K**S** 2026	$\Omega(\sqrt{\log h})$ hypercube

Warmup: vertex expansion

1. Induced-minor-free graphs
2. Two metrics at once
3. **Euclidean metrics**

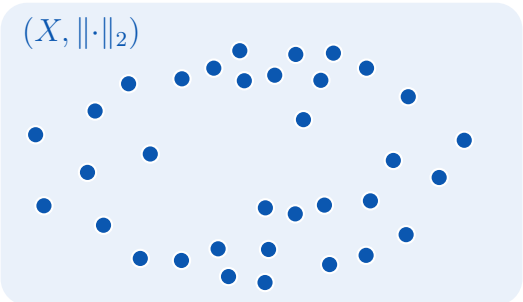
???

Euclidean metrics





Euclidean metrics





smallest $\mathrm{avd}(f)$?

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L_2^2, K_h-minor-free	$O(\sqrt{\log h})$ KS 2026	$\Omega(\sqrt{\log h})$ hypercube
Euclidean	$O(\sqrt{\log n})$ MR 2001	$\Omega\big(\sqrt{\log n/\log\log n}\big)$ MR 2001

Euclidean metrics

***Theorem S** 2026*

Every n-point $X \subset \mathbb{R}^k$ has a non-expansive $f : X \to \mathbb{R}$ with

$$\operatorname{avd}(f) = O\left(\sqrt{\frac{\log n}{\log\log n}}\right).$$

Clustered or well-spread

$$\Delta := \frac{1}{n^2} \sum_{x,y} \|x - y\|$$

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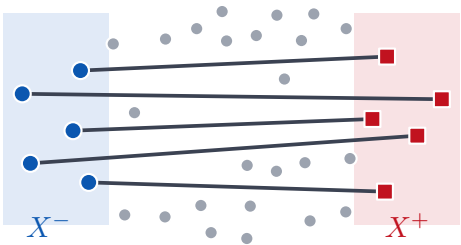
Kantorovich–Rubinstein duality



$$\max_{f \text{ non-exp.}} \left(\sum_{X^+} f - \sum_{X^-} f \right)$$

Kantorovich–Rubinstein duality

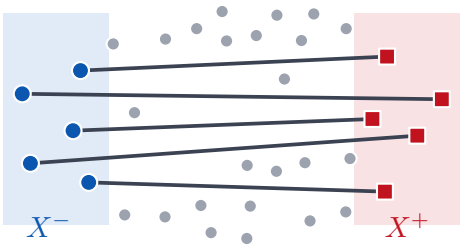


$$\max_{f \text{ non-exp.}} \left(\sum_{X^+} f - \sum_{X^-} f \right) = \min_{M} w(M)$$

M: perfect matching $X^- \leftrightarrow X^+$

$w(M) = \sum_{xy \in M} \|x - y\|$

Kantorovich–Rubinstein duality



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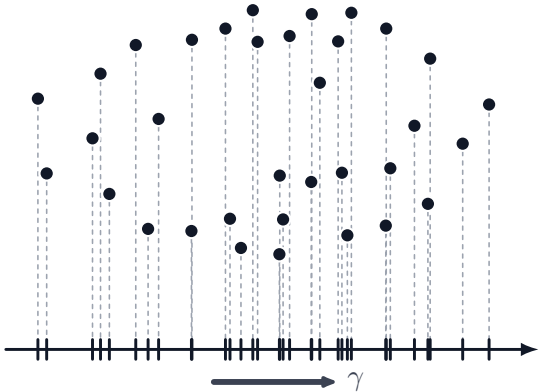
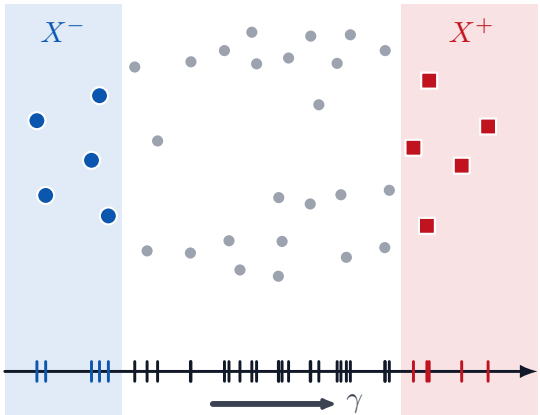
suffices: $|X^{\pm}| = n/64$ and $\min_M w(M) = \Omega\big(n\Delta\sqrt{\log\log n / \log n}\big)$

Random projection



$$\gamma \sim \mathcal{N}(0, I_k)$$

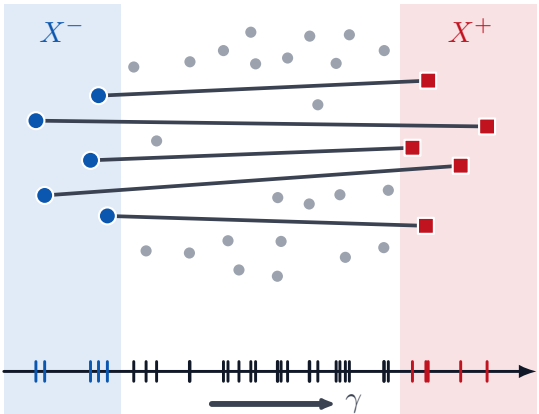
Random projection



$\gamma \sim \mathcal{N}(0, I_k)$

$X^{\pm}$: lowest / highest $n/64$

Random projection



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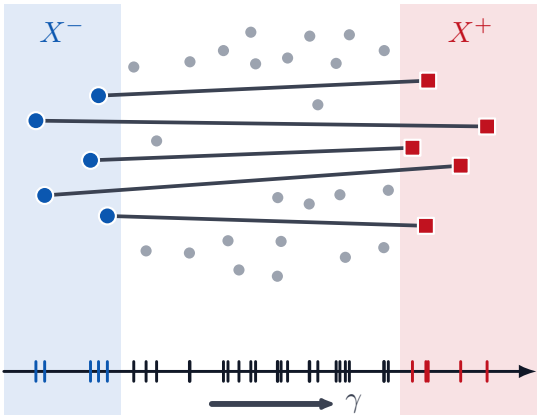

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***Corollary S** 2026*

If X is well-spread,

$$\mathbb{E}_\gamma \min_M w(M) = \Omega\left(n\Delta\sqrt{\frac{\log\log n}{\log n}}\right).$$

Vertex expansion: spectral relaxation

relaxation

$$\begin{aligned} \text{minimize} \quad & n \cdot \frac{\sum_v w_v}{\sum_{u,v} \|x_u - x_v\|^2} \\ \text{subject to} \quad & \|x_u - x_v\|^2 \leq w_u + w_v \\ & \text{for every edge } uv \end{aligned}$$

$$\lesssim \psi(G)$$

Vertex expansion: spectral relaxation

relaxation

minimize $n \cdot \dfrac{\sum_v w_v}{\sum_{u,v} \|x_u - x_v\|^2}$

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line-embedded

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suffices: $|f(u) - f(v)| \le \|x_u - x_v\|$ on edges, $\operatorname{avd}^2(f) \in O(\alpha)$

Average squared distortion

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Lemma *Naor 2014*

f non-expansive, $\operatorname{avd}(f) = D \implies$ some non-expansive h has $\operatorname{avd}^2(h) = O(D^2)$

Average squared distortion

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S 2026 + Naor: any graph: $\operatorname{avd}^2(h) \in O(\log n / \log\log n)$

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Open: an analogue for both simultaneously?

Warmup: vertex expansion

1. Induced-minor-free graphs
2. Two metrics at once
3. Euclidean metrics

???

Warmup: vertex expansion

1. Induced-minor-free graphs
2. Two metrics at once
3. Euclidean metrics
4. **The ARV L_1 embedding problem**

Edge expansion, revisited

relaxation

minimize	$n \cdot \dfrac{\sum_{uv \in E} \|x_u - x_v\|^2}{\sum_{u,v} \|x_u - x_v\|^2}$
subject to	$\|x_u - x_v\|^2$ an L_2^2 metric

$$\lesssim \phi(G)$$

Edge expansion, revisited

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$\lesssim \phi(G)$

L_1-embedded

minimize $n \cdot \dfrac{\sum_{uv \in E} \|F(u) - F(v)\|_1}{\sum_{u,v} \|F(u) - F(v)\|_1}$

subject to $F : V \to L_1$

$\approx \phi(G)$

Edge expansion, revisited

relaxation

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$\approx \phi(G)$

want: L_1-embedded $\leq O(\alpha) \cdot$ relaxation

Edge expansion, revisited

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Edge expansion, revisited

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into $\mathbb{R}$: $\alpha = \Theta(\sqrt{\log n})$ ARV 2004

Edge expansion, revisited

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into $\mathbb{R}$: $\alpha = \Theta(\sqrt{\log n})$ ARV 2004

into L_1: **?**

Open problem

Open problem

Does every n-point L_2^2 metric have a non-expansive F into L_1 with $\operatorname{avd}(F) = o(\sqrt{\log n})$?

Parallels

	Euclidean $\to \mathbb{R}$	$L_2^2 \to L_1$
existing u.b.	$O(\sqrt{\log n})$ MR 2001	

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	Euclidean $\to \mathbb{R}$	$L_2^2 \to L_1$
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Parallels

	Euclidean $\to \mathbb{R}$	$L_2^2 \to L_1$
existing u.b.	$O(\sqrt{\log n})$ **via projection** MR 2001	$O(\sqrt{\log n})$ ARV 2004

Parallels

	Euclidean $\to \mathbb{R}$	$L_2^2 \to L_1$
existing u.b.	$O(\sqrt{\log n})$ **via projection** MR 2001	$O(\sqrt{\log n})$ **into the line** ARV 2004

Parallels

	Euclidean $\to \mathbb{R}$	$L_2^2 \to L_1$
existing u.b.	$O(\sqrt{\log n})$ **via projection** MR 2001	$O(\sqrt{\log n})$ **into the line** ARV 2004
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Parallels

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Parallels

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existing u.b.	$O(\sqrt{\log n})$ **via projection** MR 2001	$O(\sqrt{\log n})$ **into the line** ARV 2004
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new u.b.	$O\big(\sqrt{\log n / \log\log n}\big)$ **S** 2026	

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new u.b.	$O\big(\sqrt{\log n / \log\log n}\big)$ **S** 2026	**?**

Review

Clustered



distance to B

$O(1)$

Review

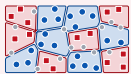
Clustered



distance to B

$O(1)$

Well-spread



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$O(\beta)$

Review

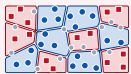
Clustered



distance to B

$O(1)$

Well-spread



padded

$O(\beta)$



separated pair

$O(\log h)$

Review

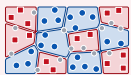
Clustered



distance to B

$O(1)$

Well-spread



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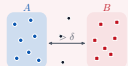
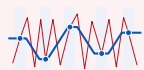
$O(\beta)$



separated pair

$O(\log h)$



project, correct

$O(\sqrt{\alpha})$

Review

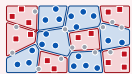
Clustered



distance to B

$O(1)$

Well-spread



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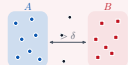
$O(\beta)$



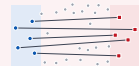
separated pair

$O(\log h)$



project, correct

$O(\sqrt{\alpha})$



project, KR duality

$O(\sqrt{\log n / \log \log n})$

Review

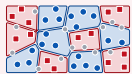
Clustered



distance to B

$O(1)$

Well-spread



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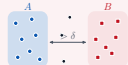
$O(\beta)$



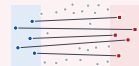
separated pair

$O(\log h)$



project, correct

$O(\sqrt{\alpha})$



project, KR duality

$O(\sqrt{\log n / \log\log n})$

Open

$K_h^{(1)}$-induced-minor-free:
padding $O(\log h)$?

Review

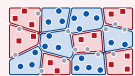
Clustered



distance to B

$O(1)$

Well-spread



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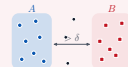
$O(\beta)$



separated pair

$O(\log h)$



project, correct

$O(\sqrt{\alpha})$



project, KR duality

$O(\sqrt{\log n / \log\log n})$

Open

$K_h^{(1)}$-induced-minor-free: padding $O(\log h)$?

Open

$L_2^2 \to L_1$: $\mathrm{avd} = o(\sqrt{\log n})$?

Review

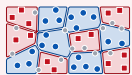
Clustered



distance to B

$O(1)$

Well-spread



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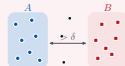
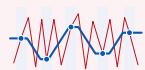
$O(\beta)$



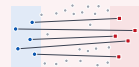
separated pair

$O(\log h)$



project, correct

$O(\sqrt{\alpha})$



project, KR duality

$O(\sqrt{\log n / \log\log n})$

Open

$K_h^{(1)}$-induced-minor-free: padding $O(\log h)$?

Open

$L_2^2 \to L_1$: $\mathrm{avd} = o(\sqrt{\log n})$?

Fin