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Graph Morphing via Orthogonal Box Drawings

Joint work with my supervisors Therese Biedl and Anna Lubiw

Presentation as part of MMath degree, 2023

Graphs, Graph Drawings, Grids

Graph: vertex set V, edge set $E \subset \binom{V}{2}$ (no multiedges, no self-loops)

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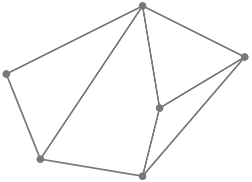
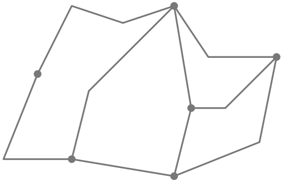
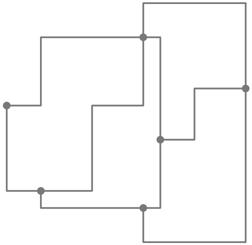
Graph drawing: Representation of a graph on a plane
Planar graph drawing: Graph drawing where edges do not intersect



Vertices: Points
Edges: Line segments



Vertices: Points
Edges: Polylines



Vertices: Points
Edges: Orthogonal polylines

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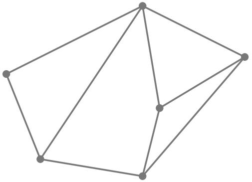

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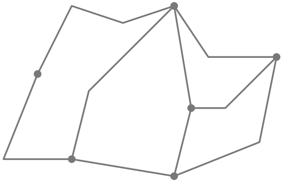

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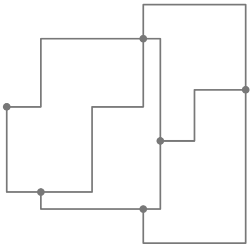
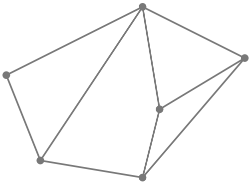


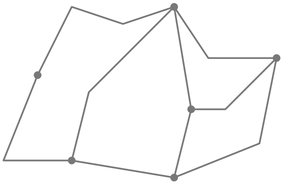
Vertices: Points
Edges: Orthogonal polylines

Our graphs will always be planar graphs (planar drawing exists).

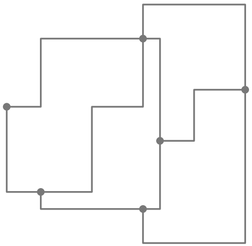
Point Drawings



Planar straight-line drawing



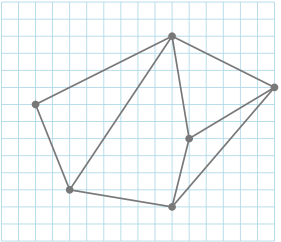
Planar poly-line drawing



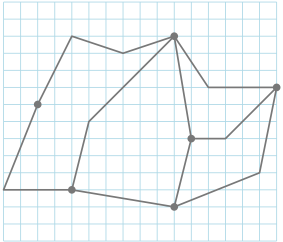
Planar orthogonal point drawing

Vertices are points.

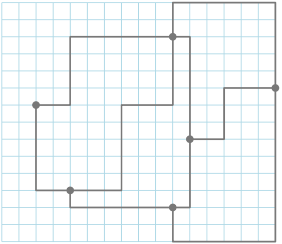
Point Drawings



Planar straight-line drawing



Planar poly-line drawing

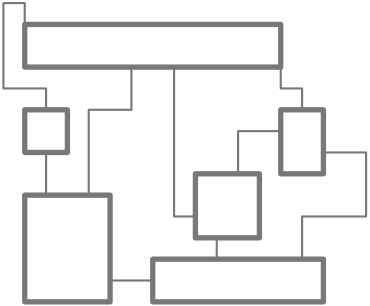


Planar orthogonal point drawing

Vertices are points.

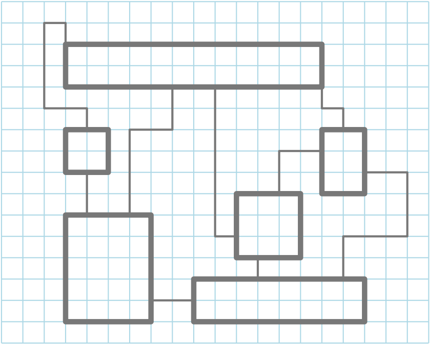
Can be drawn on a grid.

Non-Point Drawings



Planar orthogonal box drawing
Vertices: Axis-aligned rectangles
Edges: Orthogonal polylines

Non-Point Drawings

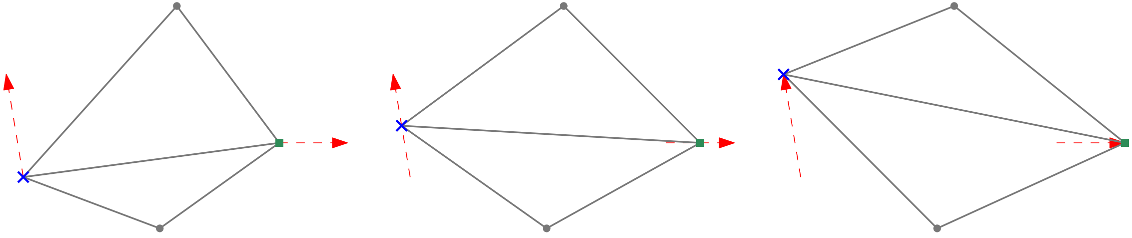


Planar orthogonal box drawing
Vertices: Axis-aligned rectangles
Edges: Orthogonal polylines

Can be drawn on a grid.

Graph Morphing

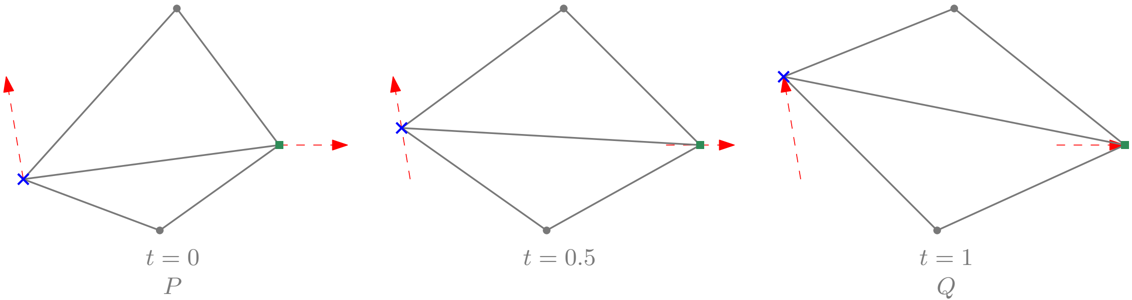
Morph: Continuously deform between drawings



$t = 0$ P $t = 0.5$ $t = 1$ Q

Graph Morphing

Morph: Continuously deform between drawings



Linear morph: Linearly interpolate vertex (and other) locations

Graph Morphing

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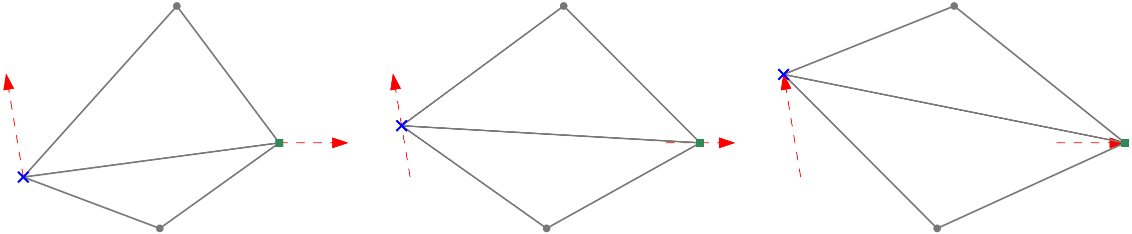

$t = 0$
P

$t = 0.5$

$t = 1$
Q

Linear morph: Linearly interpolate vertex (and other) locations

Morphs are always reversible!

Planarity-Preserving Morphs

Planarity-Preserving Morph: At all times t, the "interpolated" drawing is also planar.

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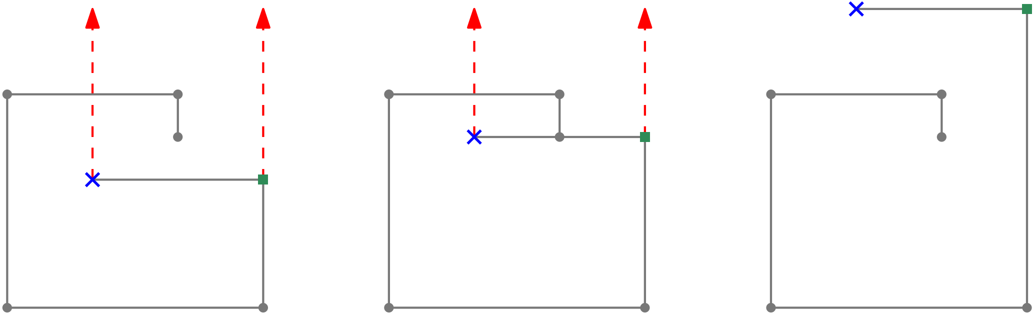
Non-planarity-preserving morph:

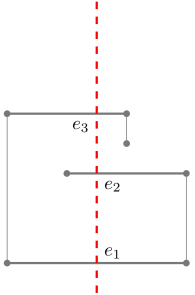


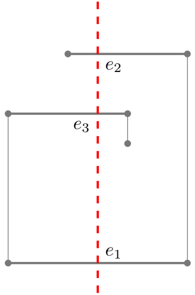
$t = 0$ $t = \frac{1}{4}$ $t = 1$

Unidirectional Linear Morphs

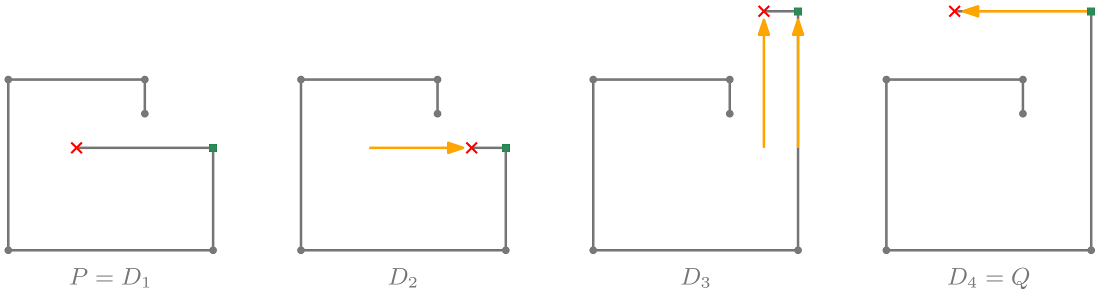
Unidirectional linear morph: All movement directions are parallel.

Fact (Alamdari et al., Kleist et al.): Unidirectional linear morphs are planarity-preserving if and only if every line parallel to the direction of movement has the same intersection order in both drawings.



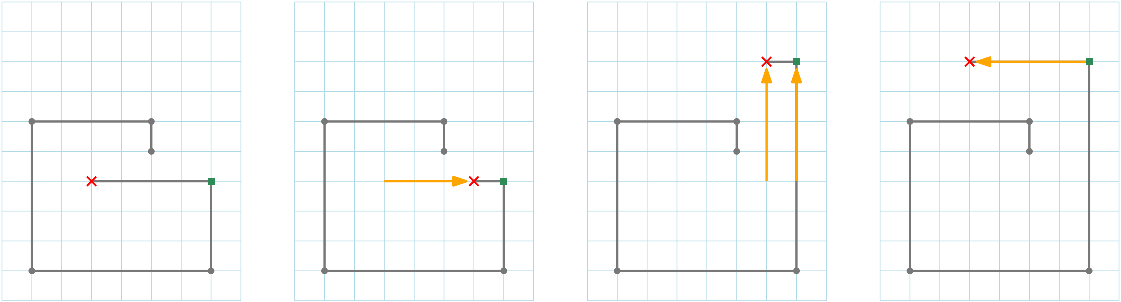


Linear Morphs Sequences



The above are the **explicit intermediate drawings**.
Entire morph is represented by the sequence D_1, D_2, D_3, D_4.

Linear Morph Sequences on a Grid



$P = D_1$ D_2 D_3 $D_4 = Q$

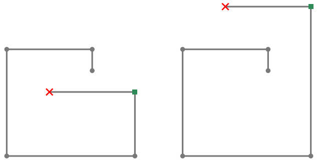
Explicit drawings are on a grid.
Implicit (interpolated) drawings are not.

Linear Morphs Sequences that Add/Remove Bends

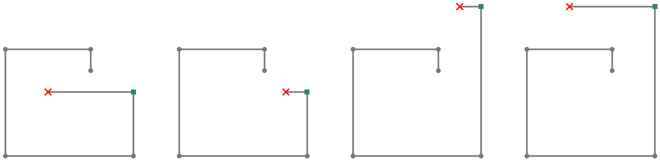
Degenerate bend: Bend that "isn't used" (coincident or 180° angle).
Equivalent drawings: Drawings that differ only by degenerate bends.

The Linear Morph Problem

Input:
'Compatible' pair of drawings (labelled)



Output:
Planarity-preserving linear morph sequence (list of drawings)



Objectives: Numerous!

Previous Results

Input: ('Compatible') pair of drawings
Output: Planarity-preserving linear morph sequence

	Graph/Drawing Class	Num linear morphs	Grid-size side-length	Bends per edge	Comput. Model	Time Complexity
Alamdari et al. (2017)	Straight-line	$O(n)$	Expo.	0	Powerful	$O(n^3)$
Klemz (2021)	Straight-line	$O(n)$	Expo.	0	Powerful	$O(n^2 \log n)$

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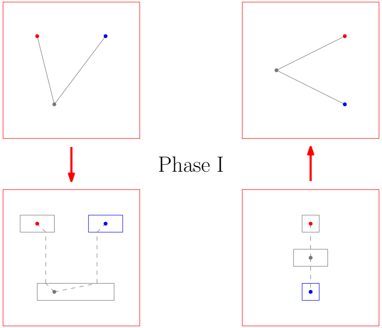
Biedl et al. (2013)	Connected Orthogonal	$O(n^2)$	$O(n)$	$O(n)$	Word RAM	Polynomial
Van Goethem et al. (2022)	Orthogonal	$O(n)$	Polynomial	$O(1)$	Word RAM	Polynomial
This work (main method)	Connected Ortho-Box	$O(n)$	$O(n)$	$O(1)$	Word RAM	$O(n^2)$

Grid size assumes input fits on the same grid.

Above table is not comprehensive.

High-Level Overview

First: Reduce straight-line drawing morphing problem → orthogonal box drawing morphing problem.



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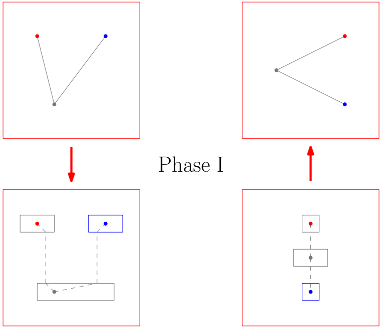
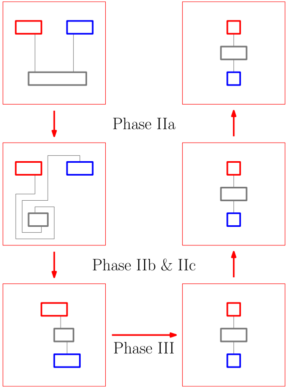
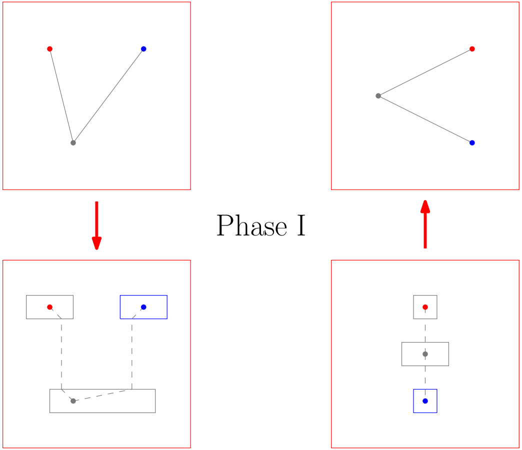


Second: Solve orthogonal box drawing morphing problem using (improved) techniques for orthogonal point drawing morphing problem.



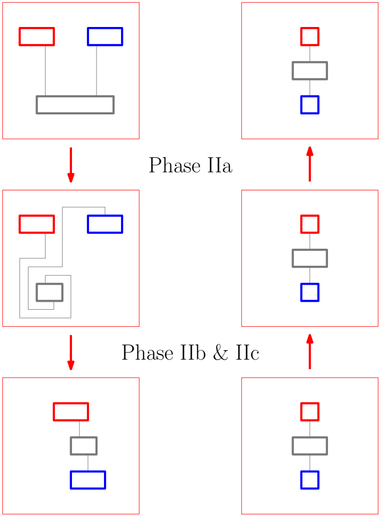
Phase I

High-level: Reduce to box drawing morphs.
Need to do a morph, and give a reduction.



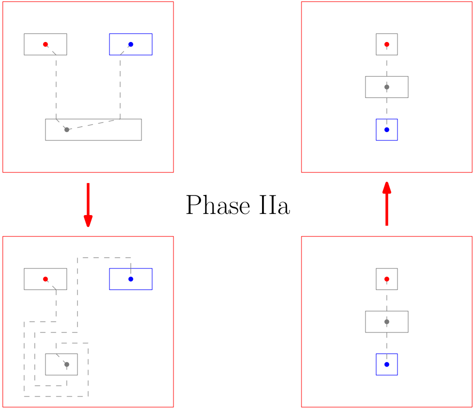
Phase II

High-level: Reduce to parallel box drawing morphs (only lengths differ). Only need to morph (not a different drawing type).



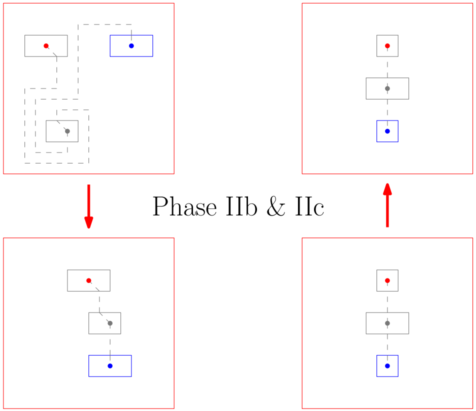
Phase IIa

High-level: Move ports. Add bends to do so.



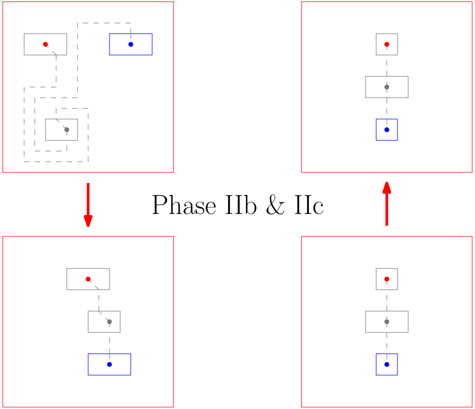
Phase IIb

High-level: Do some (global) analysis on the edges.



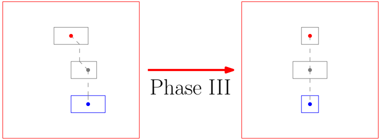
Phase IIc

High-level: Use analysis to get rid of bends.

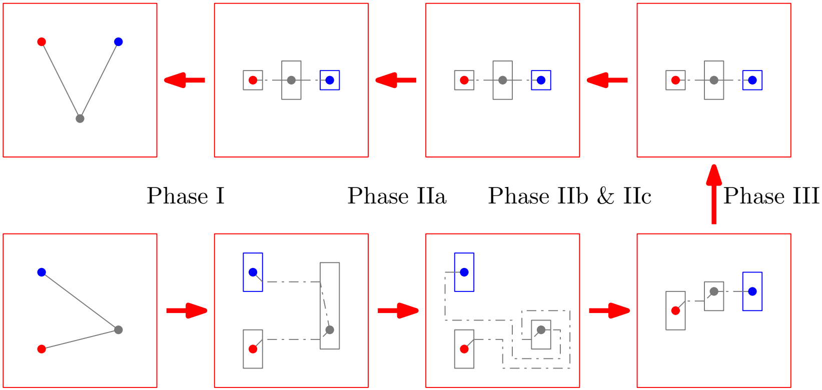


Phase III

High-level: Use black-box result to morph parallel orthogonal box drawings (i.e., adjust lengths).



The Phases—All Together



An example of all phases on a very simple drawing.

- Phase I: Reduce to boxes.
- Phase IIa: Edit ports.
- Phase IIb: Analyze edges.
- Phase IIc: Globally 'unify'.
- Phase III: Morph unified drawings.

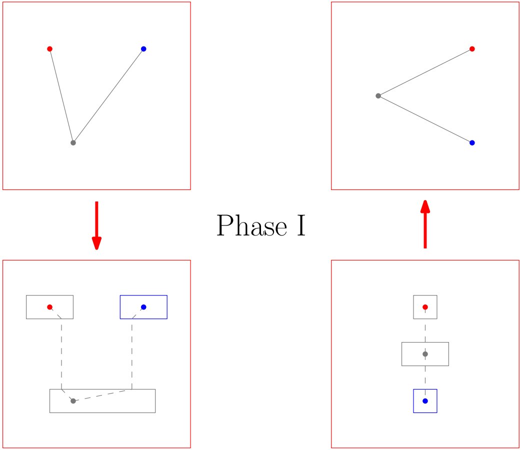
And now, details!

Phase I

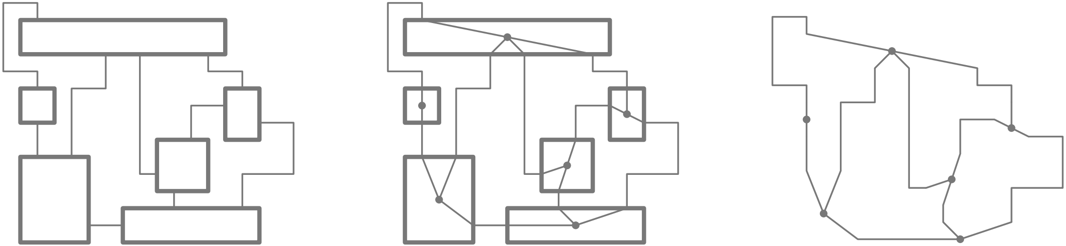
High-level: Reduce to box drawing morphs.
Need to do a morph, and give a reduction.

Phase I Overview

- **Input:** Straight-line drawing pair
- **Output:** Box drawing pair
- Also need a reduction (morphing box drawings ≊ morphing point drawings).

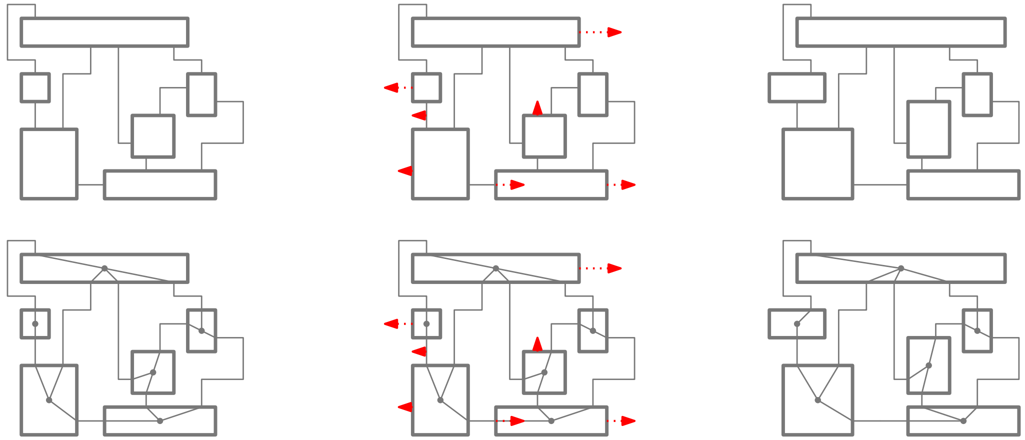


Reduction: Admitted Drawings (1)



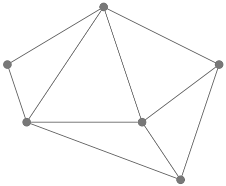
Orthogonal box drawing | Both | Admitted poly-line drawing

Reduction: Admitted Drawings (2)

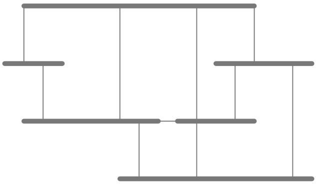


Morph of orthogonal box drawings $\Longrightarrow$ morph of admitted drawings

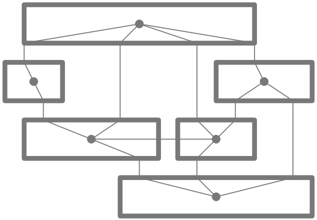
Computing Box Drawings: Visibility Representations as an Intermediary



A planar straight-line drawing P.

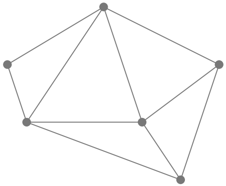


A visibility representation that can be computed from P.



An orthogonal box drawing, and corresponding admitted drawing P', which can both be computed from P.

Computing Box Drawings: Visibility Representations as an Intermediary



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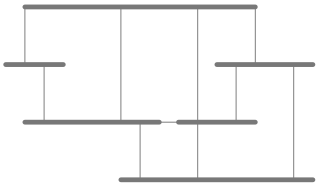
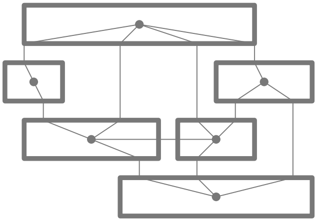


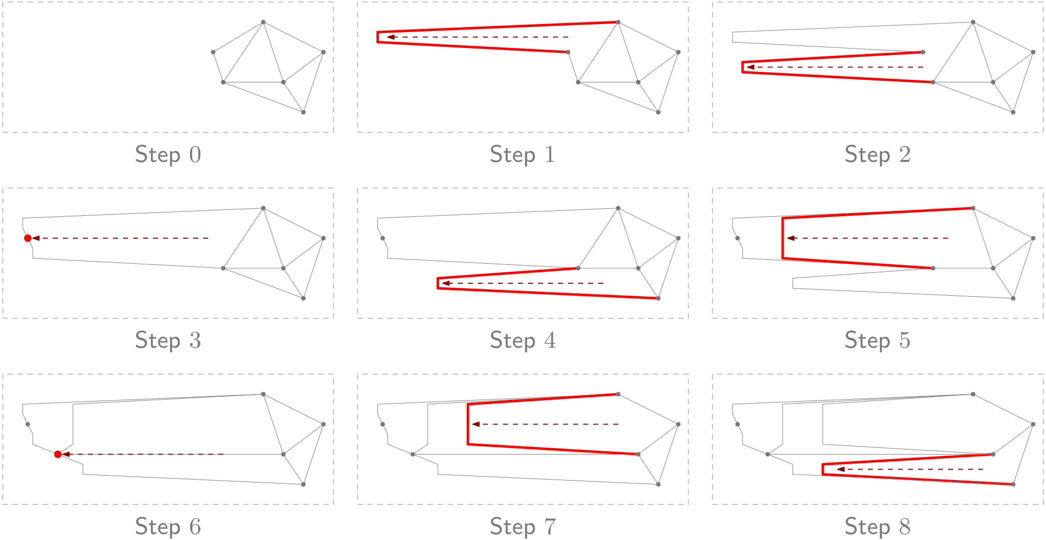
A visibility representation that can be computed from P.



An orthogonal box drawing, and corresponding admitted drawing P', which can both be computed from P.

How do we actually perform a morph?

Morphing from a straight-line to an admitted drawing: Method



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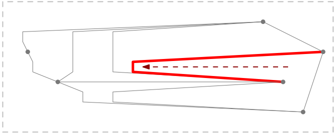
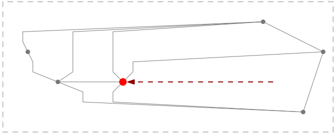
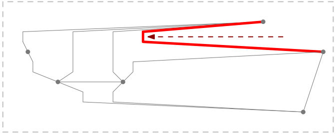


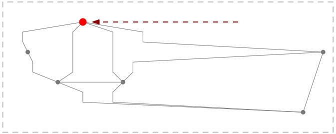
Step 9



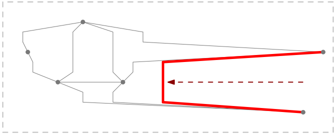
Step 10



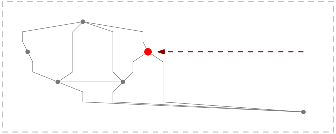
Step 11



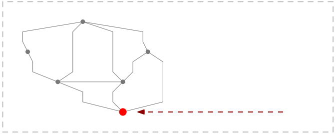
Step 12



Step 13



Step 14



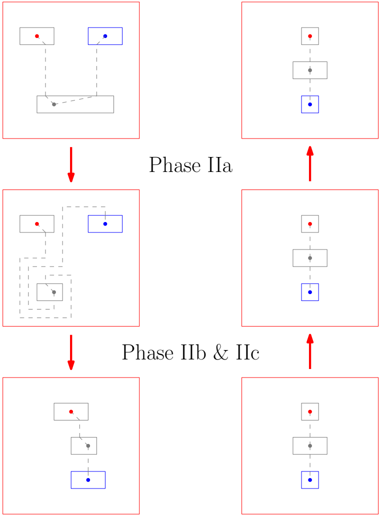
Step 15

Phase II

High-level: Reduce to parallel box drawing morphs (only lengths differ). Only need to morph (not a different drawing type).

Phase II Overview

- **Input:** Orthogonal box drawing pair
- **Output:** Parallel orthogonal box drawing pair (for each edge: same port locations, same sequence of turns)
- Substeps:
  - Phase IIa: Adjust port locations
  - Phase IIb: Global analysis $\mapsto$ instructions
  - Phase IIc: Instructions $\mapsto$ local changes



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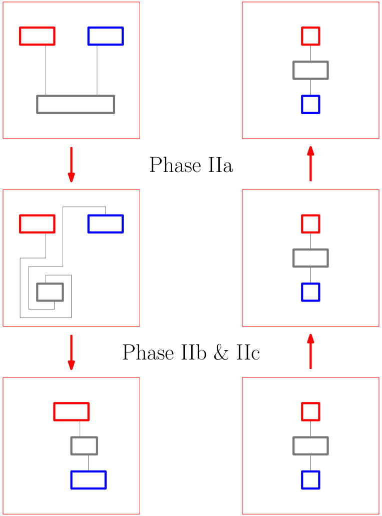
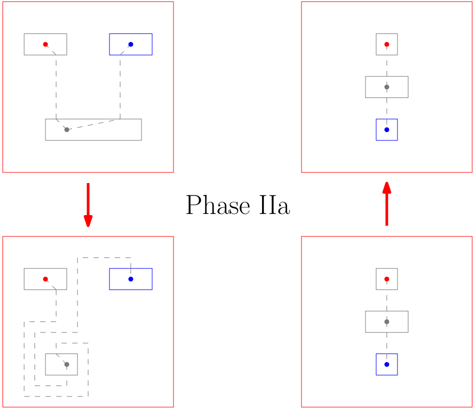


Phase IIa

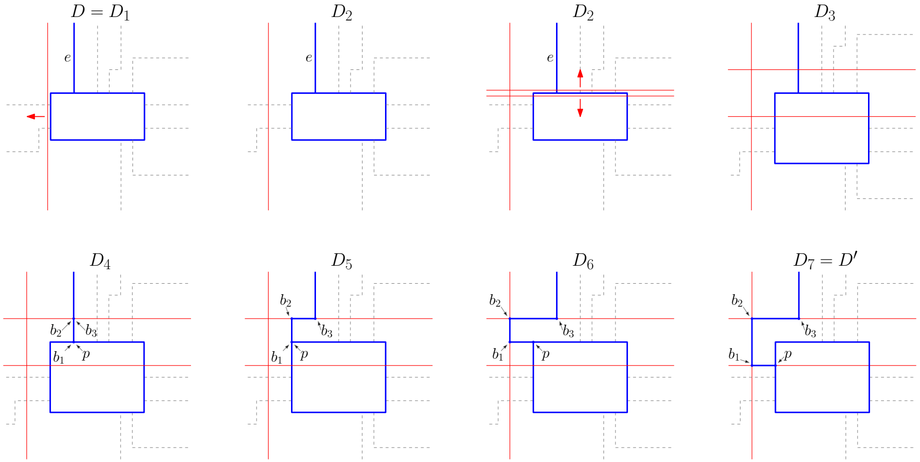
High-level: Move ports. Add bends to do so.

Phase IIa Overview

- **Input:** Orthogonal box drawing pair
- **Output:** Port-aligned orthogonal box drawing pair (same relative port locations)

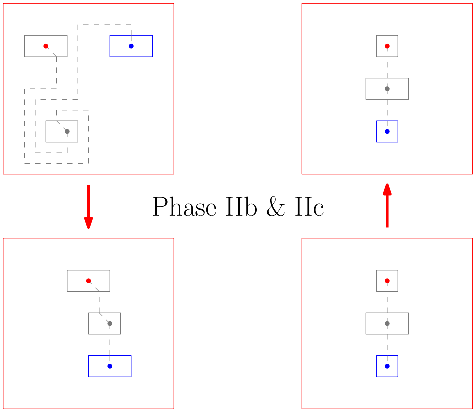


Moving Ports around Corners



Phase IIb

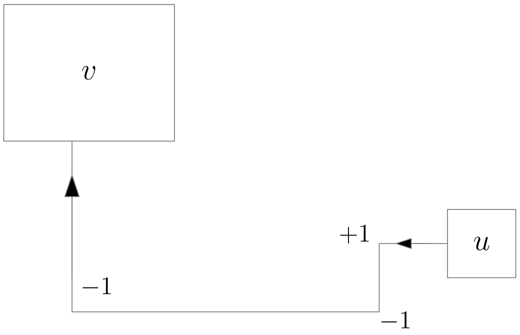
High-level: Do some (global) analysis on the edges.



Phase IIb Overview

- **Input:** Port-aligned orthogonal box drawing pair
- **Output:** "Instructions"

Spirality

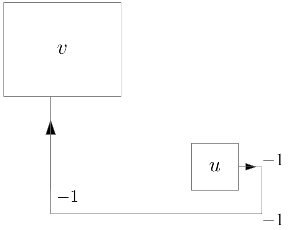


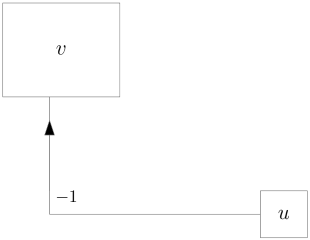
Spirality of the edge uv (oriented u to v): -1.

Difference in Spirality (1)

Difference in spirality of the edge uv (oriented u to v): -2.

Difference in Spirality (1)

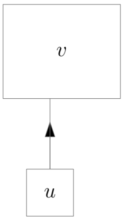


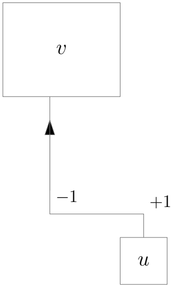


Difference in spirality of the edge uv (oriented u to v): -2.

Goal: Reduce this to zero.

Difference in Spirality (2)

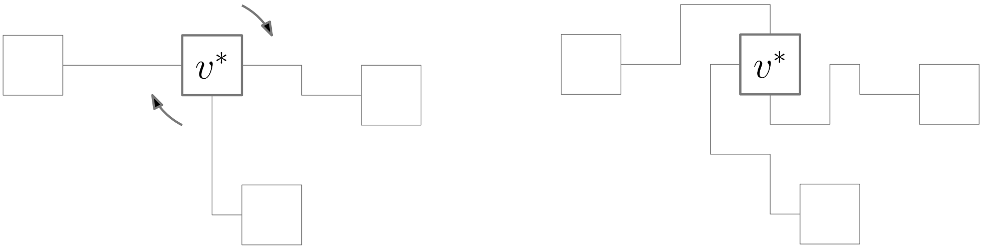




Difference in spirality of the edge uv (oriented u to v): 0.

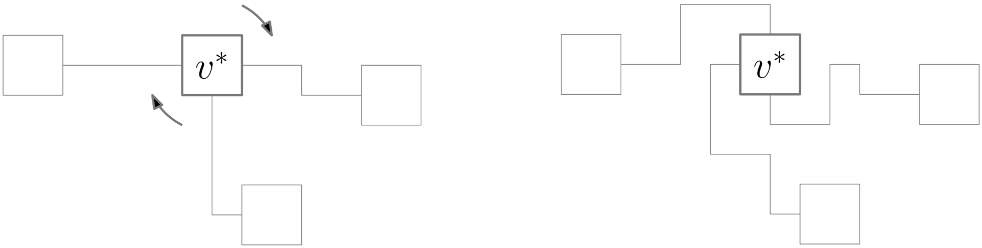
Goal: Reduce this to zero.

Twists (High-Level)



Spirality changes! Net turns are added.
Don't know how to compute these drawings yet (happens in Phase IIc).

Twists (High-Level)



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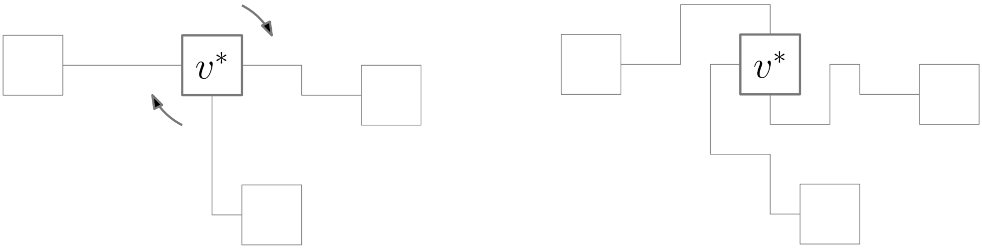
Similar to a result by Biedl et al.: Exists some number/direction of twists for each vertex so that difference in spirality becomes zero everywehere. This number is $O(n)$ for each vertex.

Twists (High-Level)



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Similar to a result by Biedl et al.: Exists some number/direction of twists for each vertex so that difference in spirality becomes zero everywehere. This number is $O(n)$ for each vertex.

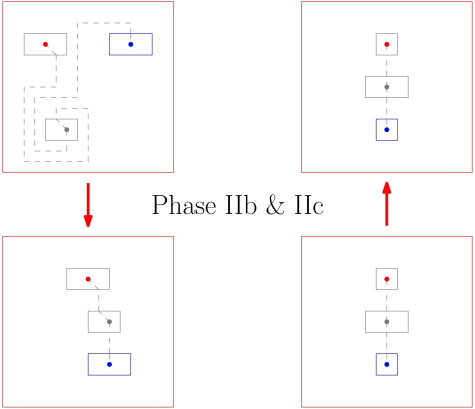
Key difference/contribution: We use simultaneous twists, so only $O(n)$ operations needed.

Phase IIc

High-level: Use analysis to get rid of bends.

Phase IIc Overview

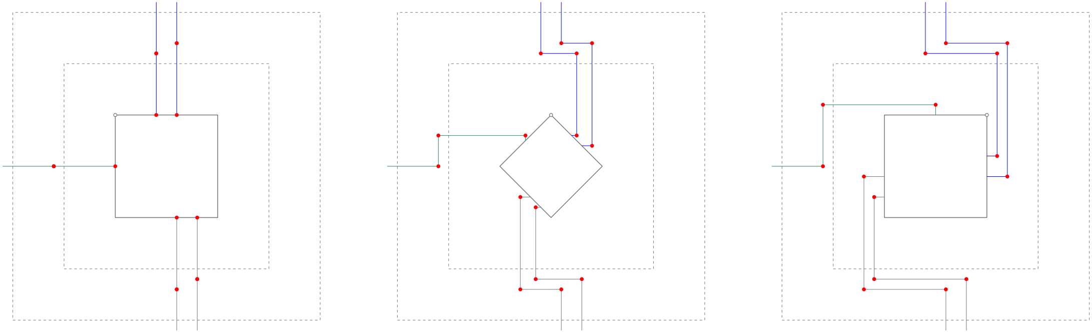
- **Input:** Port-aligned orthogonal box drawing pair, simultaneous twist instructions
- **Output:** Parallel orthogonal box drawings
- Two components:
  - Perform twists
  - Obtain canonical drawings



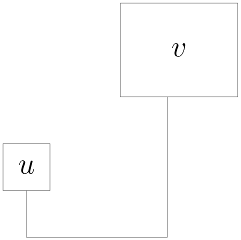
Twists

Two steps:

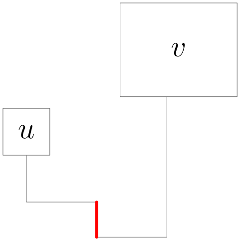
- "Prepare" drawing (make boxes square, well-spaced out)
- Twist everything simultaneously



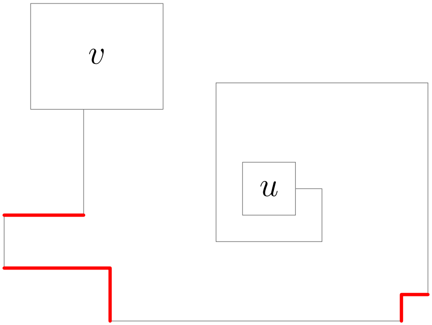
Simplification/Canonical form: Zig-Zags



No zig-zags.



One (vertical) zig-zag.

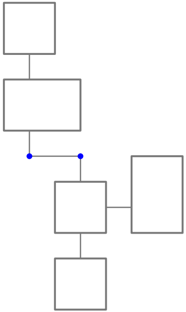


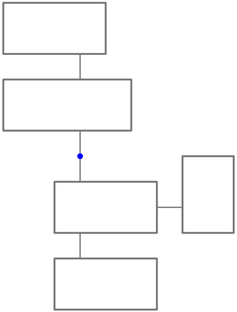
Five zig-zags, three horizontal and two vertical.

We want to remove zig-zags.

Simplification/Canonical form: Removing a Single Zig-Zag (1)

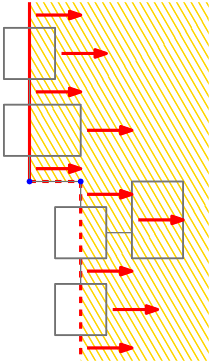
Method by Biedl et al.:

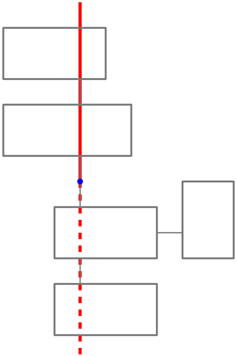




This is a unidirectional morph.

Simplification/Canonical form: Removing a Single Zig-Zag (2)





Push each thing over if it lies to the right of the divider.

Simplification/Canonical form: Removing a Single Zig-Zag (2)

Push each thing over if it lies to the right of the divider.

Problem: Requires a morph for each zig-zag (want $O(1)$ morphs for all zig-zags).

Simplification/Canonical form: Removing a Single Zig-Zag (2)

Push each thing over if it lies to the right of the divider.

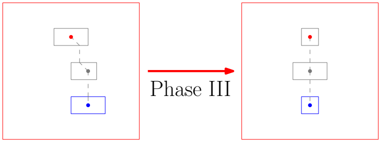
Problem: Requires a morph for each zig-zag (want $O(1)$ morphs for all zig-zags).
Solution/new contribution: $O(1)$ morphs suffice, even on a grid (skipping details).

Phase III

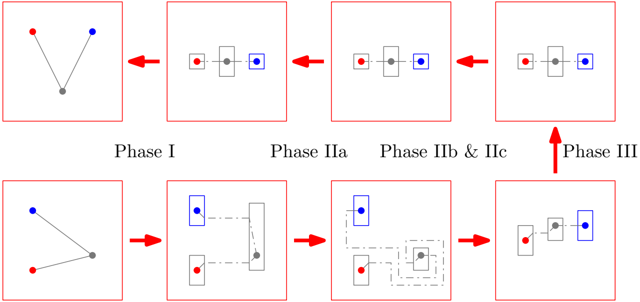
High-level: Use black-box result to morph parallel orthogonal box drawings (i.e., adjust lengths).

Phase III Overview

- **Input:** Parallel orthogonal box drawing pair.
- **Output:** Linear morph sequence.
- **Methodology:** Appeal to black-box result by Biedl et al.. It requires connectivity.
  - Essentially, add edges to both drawings (and simplify again) until every face is a rectangle.



	Graph/drawing class	Num. linear morphs	Grid size side-length	Bends per edge	Time complexity
Main result	Connected	$O(n)$	$O(n)$	$O(1)$	$O(n^2)$
Main method	Connected Ortho-Box	$O(n)$	$O(n)$	$O(1)$	$O(n^2)$



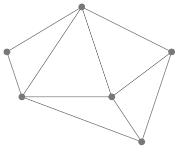
Open problems:

- Seemingly approachable: Unidirectional morphs only, max 4 bends, max 2 bends, disconnected graphs.
- Big: No bends.

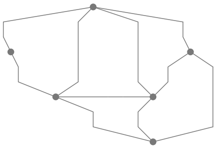
Fin.

Morphing from a straight-line to an admitted drawing: Brainstorming (1)

Have: a planar straight-line drawing P, an orthogonal box drawing D with an admitted drawing P'.
Want: Morph from P to P'. Bends need to be added.

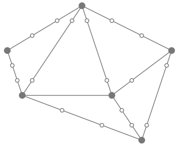


P



P'

- Idea 1: Use same y-coordinate
- Problem: Not integer coordinates



Morphing from a straight-line to an admitted drawing: Brainstorming (2)

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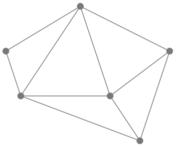
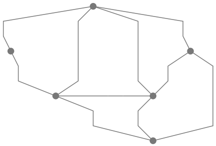


P



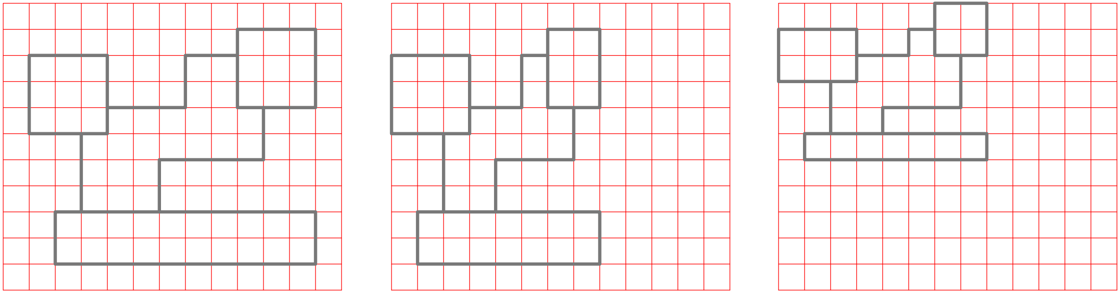
P'

- ~~Idea 1: Use same y-coordinate~~
- Idea 2: Make them coincident with the vertex
- Possible problem: Not a unidirectional morph (complicated movement).
- Alleviation: Perform the morph on one vertex/edge at a time.

Compressions

Want to be able to bring a drawing to an $O(n) \times O(n)$ grid from an arbitrarily sized grid (where the constant is independent of the initial grid size).

Idea: Sort x-coordinates.



This is a unidirectional morph.

Simplification—Removing all Horizontal Zig-Zags (High-level)

Each problem has a different solution:

- Requires a morph for each zig-zag (want $O(1)$ morphs for all zig-zags).
  - Van Goethem et al.: A single morph suffices for many (disjoint) horizontal zig-zags.

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    - Slow time complexity.

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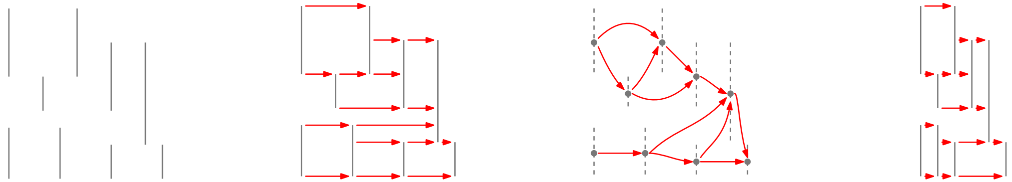
  - Van Goethem et al.: A single morph suffices for many (disjoint) horizontal zig-zags. Two issues with their solution:
    - Uses a large grid.
    - Slow time complexity.
- Requires $O(n)$ time for each zig-zag (want $O(n)$ time for all zig-zags).
  - Use circuit layout compaction!

Simplification—Circuit Compaction

Goal: Compress vertical line segments.

Solution by Doenhardt and Lengauer:

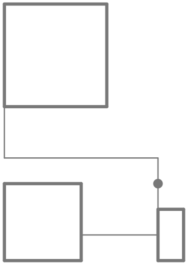


(1) Input (2) Trapezoidal Map (3) Trapezoidal Graph (4) Result from topological sort

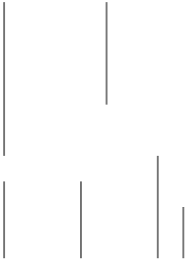
Important note:
Last step of Doenhardt and Lengauer's algorithm only needs y-coordinates and trapezoidal graph.

Simplification—Circuit Compaction for Box Drawings

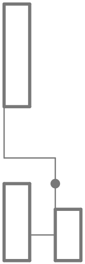
Goal: Compress a box drawing (again).



A box drawing C



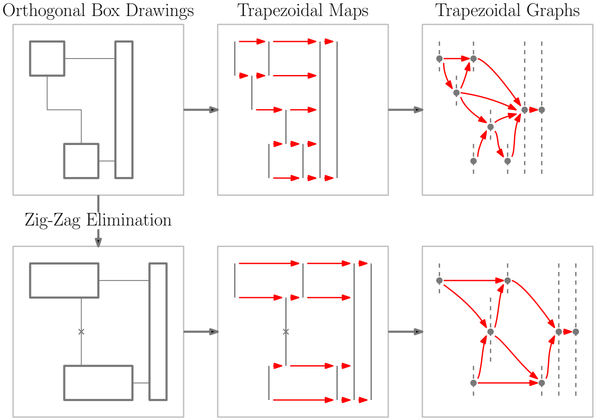
A set of maximal vertical line segments $L(C)$ covering C



The compressed drawing C'

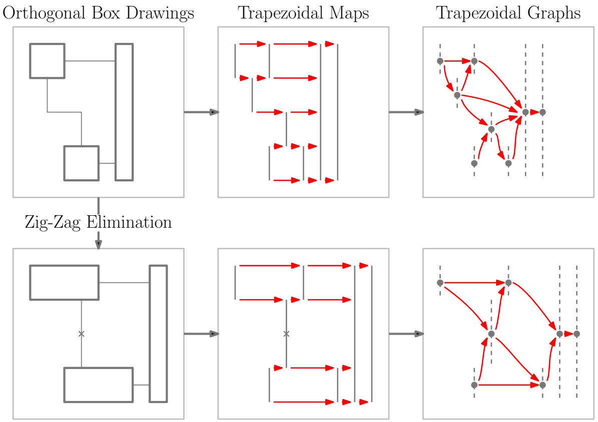
Side note: Doing this in $O(n)$ time requires connectivity (via an algorithm by Chazelle for trapezoidal maps of simple polygons).

Simplification—Zig-Zag Elimination and Circuit Compaction



Takeaway: The changes to the trapezoidal graph are local to the zig-zag being eliminated.

Simplification—Zig-Zag Elimination and Circuit Compaction



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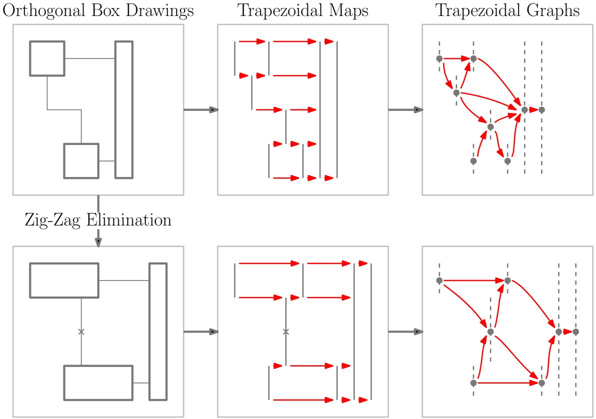
Recall: Last step of Doenhardt and Lengauer's algorithm only needs y-coordinates and trapezoidal graph.

Simplification—Zig-Zag Elimination and Circuit Compaction



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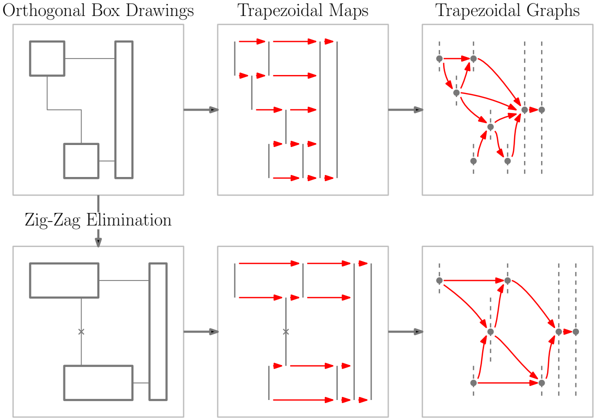
Idea: Compute only the trapezoidal graph after a sequence of zig-zag eliminations.

Simplification—Zig-Zag Elimination and Circuit Compaction



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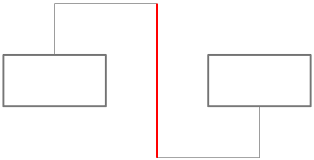
Recall: Last step of Doenhardt and Lengauer's algorithm only needs y-coordinates and trapezoidal graph.

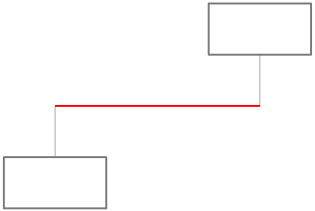
Idea: Compute only the trapezoidal graph after a sequence of zig-zag eliminations.

Final result: Can remove all horizontal zig-zags in one linear morph, in $O(n)$ time.

Simplification—Eliminating All Zig-Zags

Eliminating all horizontal zig-zags $\neq$ eliminating all zig-zags:





Simplification—Eliminating All Zig-Zags

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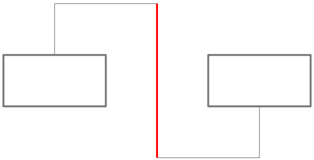

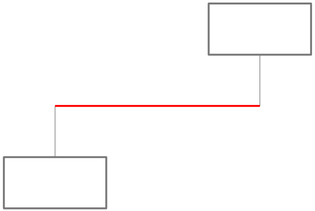

Eliminating all horizontal (and then vertical) zig-zags does reduce the number of bends per edge (unless there are no zig-zags).

Simplification—Eliminating All Zig-Zags

Eliminating all horizontal zig-zags $\neq$ eliminating all zig-zags:

Eliminating all horizontal (and then vertical) zig-zags does reduce the number of bends per edge (unless there are no zig-zags).

Idea: Since $O(1)$ bends per edge is maintained, only need to do $O(1)$ simultaneous eliminations to eliminate all zig-zags.

Phase II High-Level