Polynomial-Time Algorithms for Contiguous Art Gallery and Related Problems

Ahmad Biniaz, Anil Maheshwari, Magnus Christian Ring Merrild, Joseph S.B. Mitchell, Saeed Odak, Valentin Polishchuk, Eliot W. Robson, Casper Moldrup Rysgaard, Jens Kristian Refsgaard Schou, Thomas Shermer, Jack Spalding-Jamieson, Rolf Svenning, Da Wei Zheng

SoCG June 2025

Combined work

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The Contiguous Art Gallery Problem is Solvable in Polynomial Time,

- Merrild, Rysgaard, Schou & Svenning

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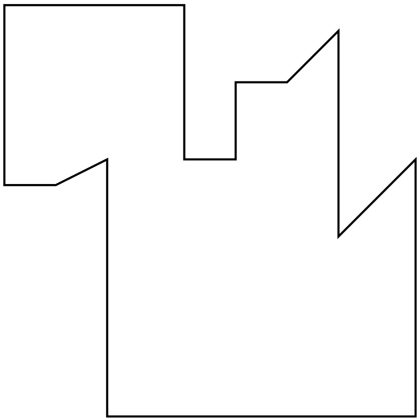
- Robson, Spalding-Jamieson & Zheng

Art Gallery



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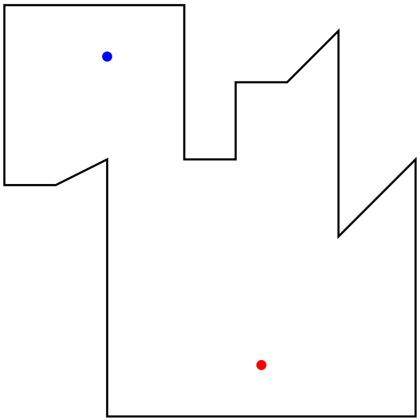

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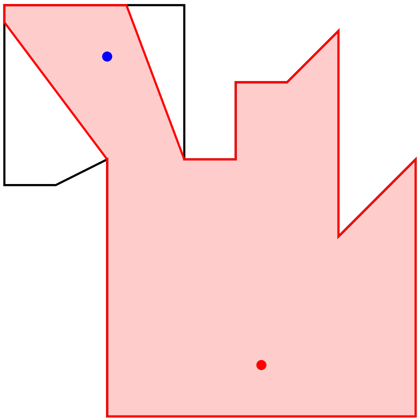

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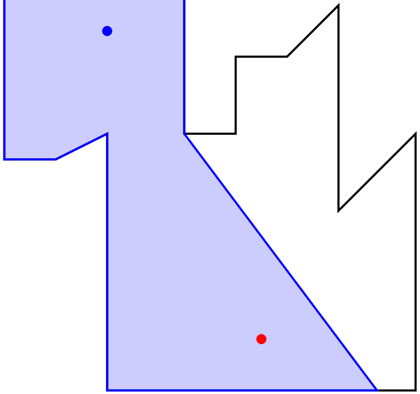

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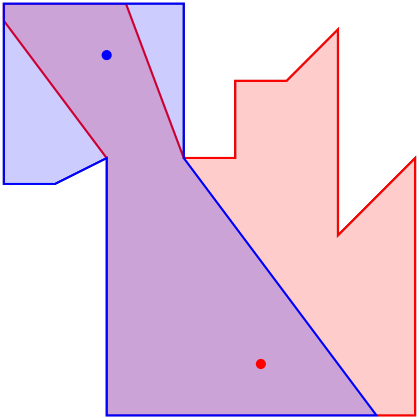

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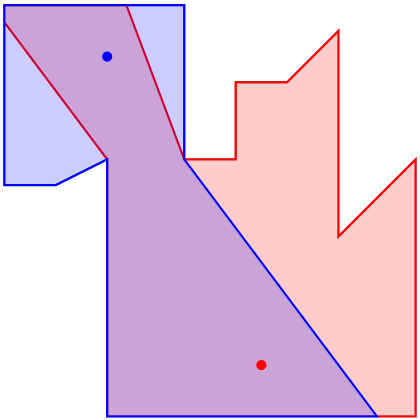
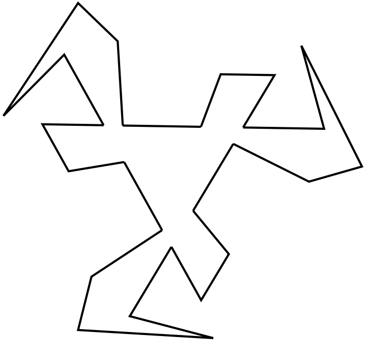


$\exists\mathbb{R}$-complete

[Abrahamsen, Adamaszek Miltzow, 2021]

Boundary Guarding



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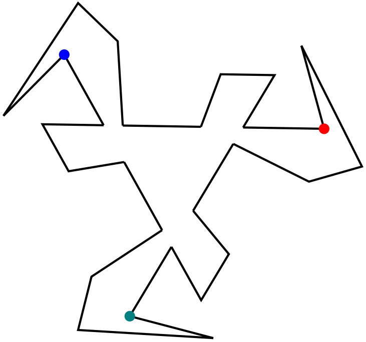

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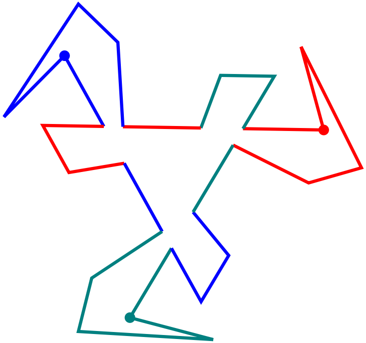

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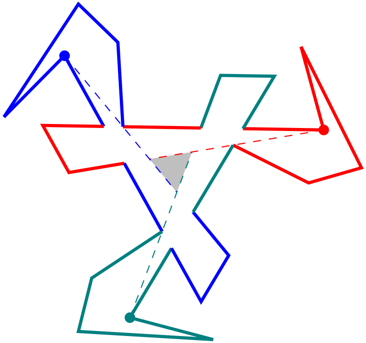

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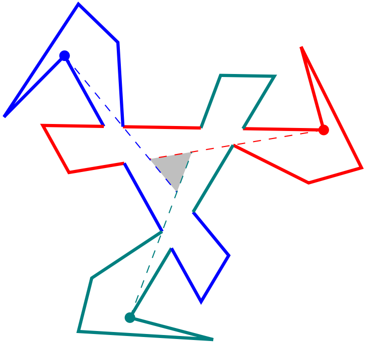

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[Stade, SoCG25]

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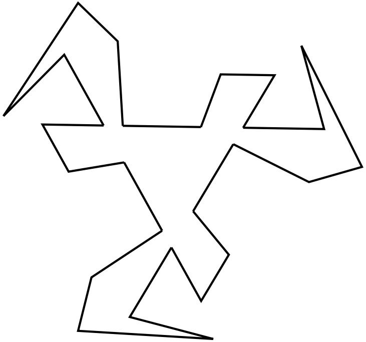

Goal: Guard boundary (few guards)

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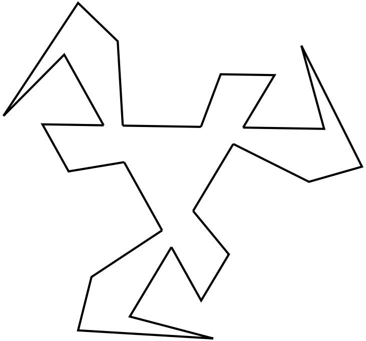

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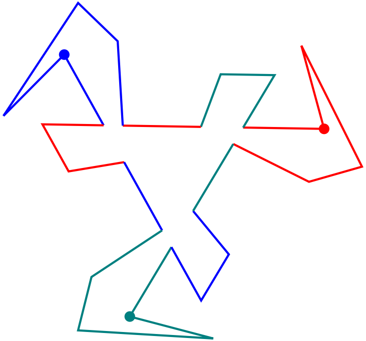

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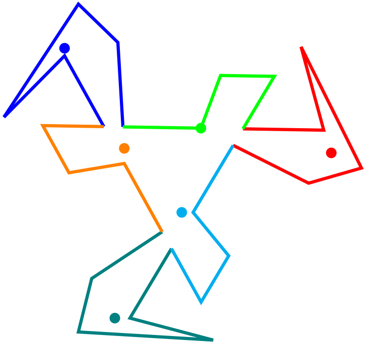

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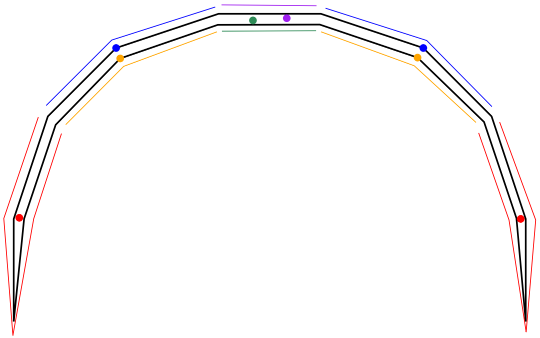

The Greedy Algorithm

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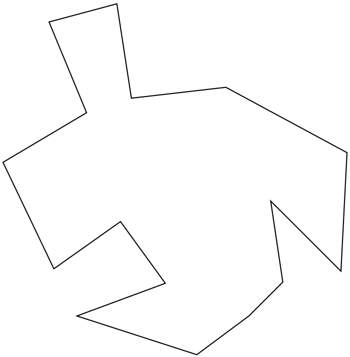

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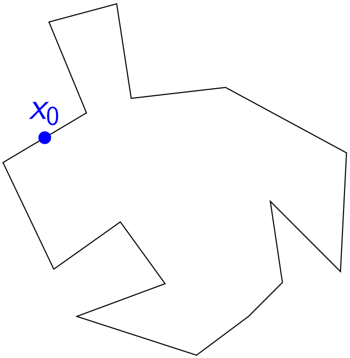

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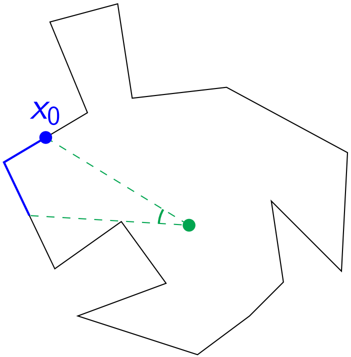

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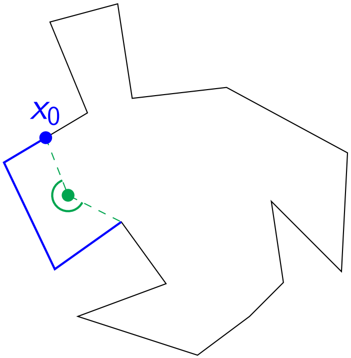

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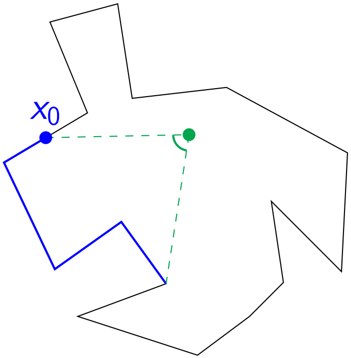

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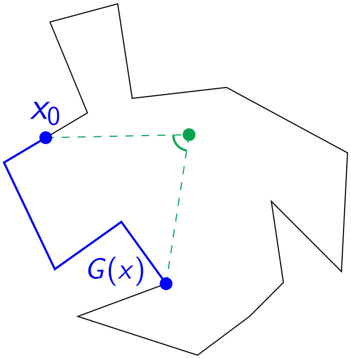

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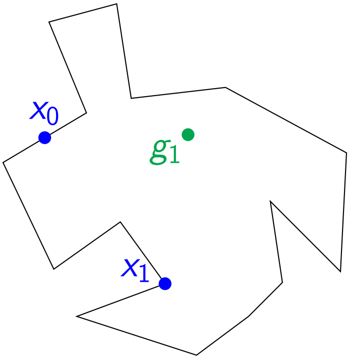

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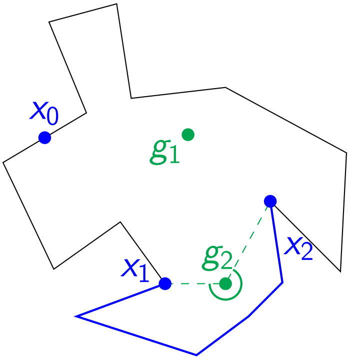

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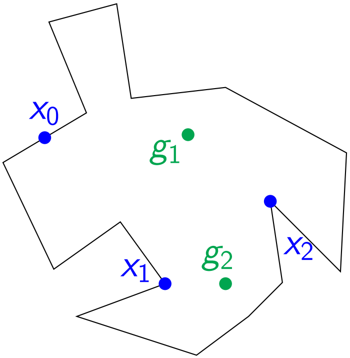

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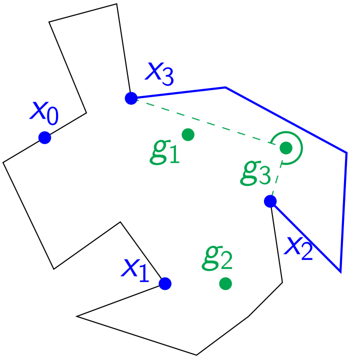

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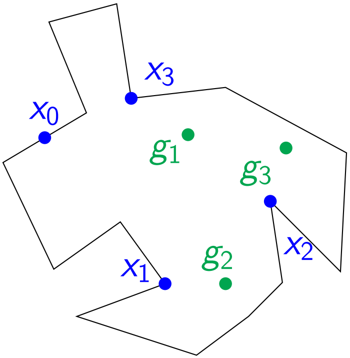

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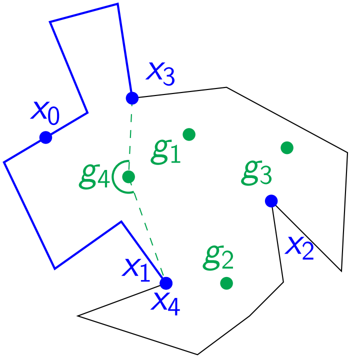

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Exact solution attainable?

Exact algorithms

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Three different algorithms/methods.

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- Method 2: Select good candidate starting points
- Method 3: Try every starting point simultaneously

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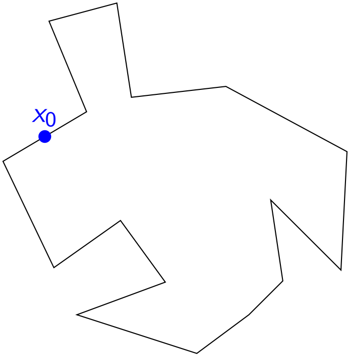

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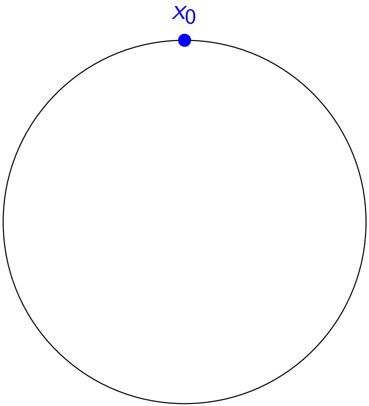

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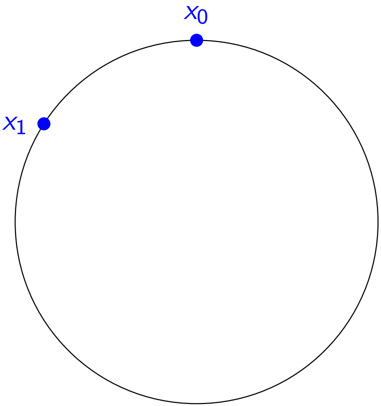

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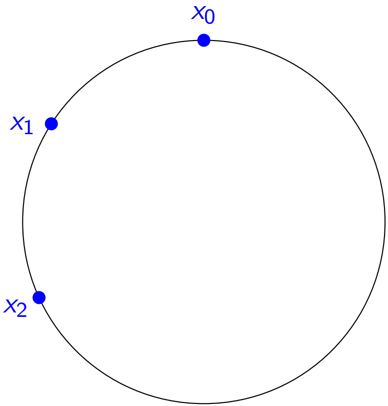

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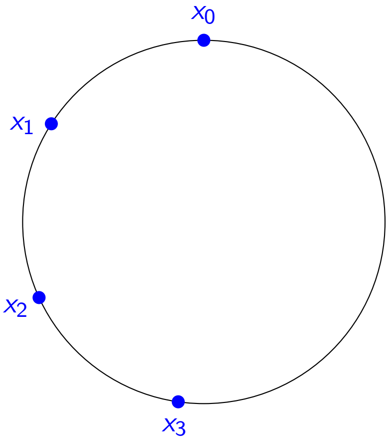

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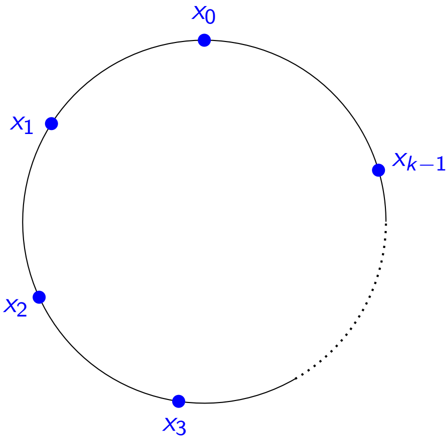

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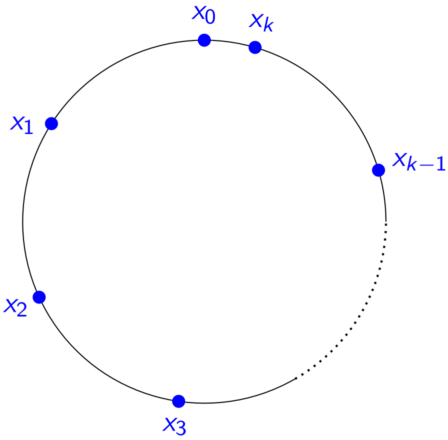

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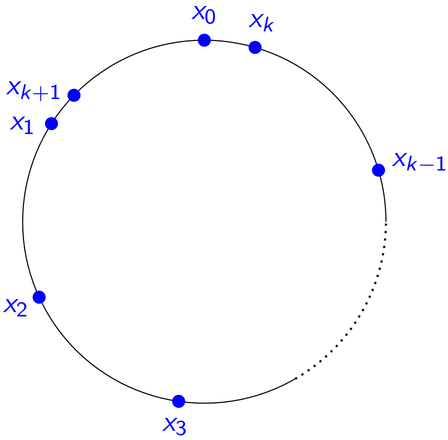

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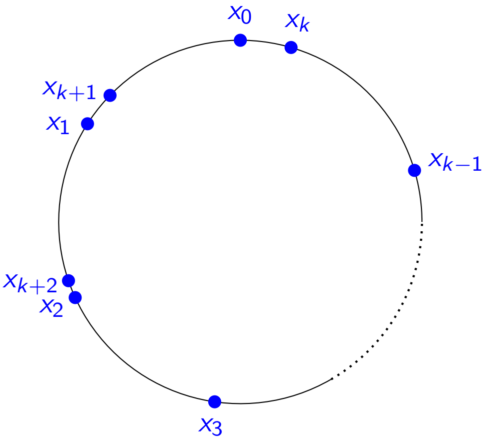

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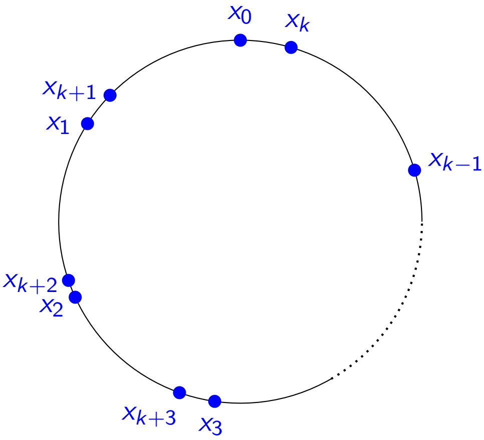

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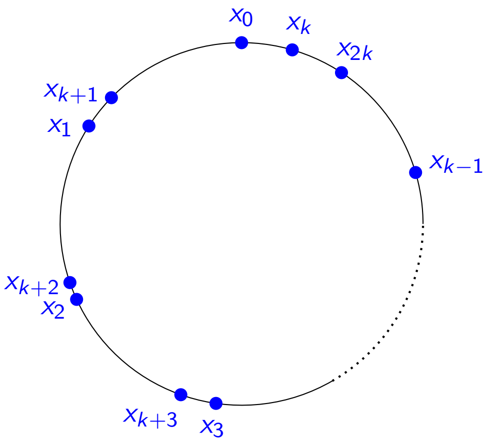

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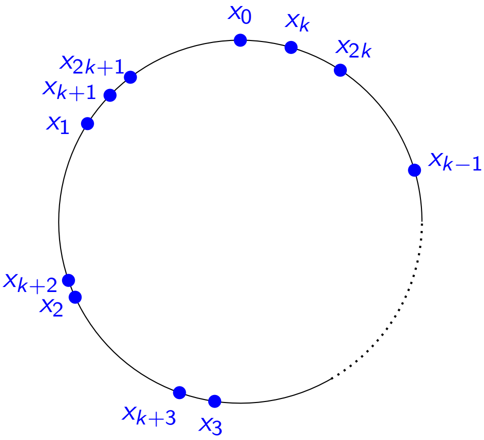

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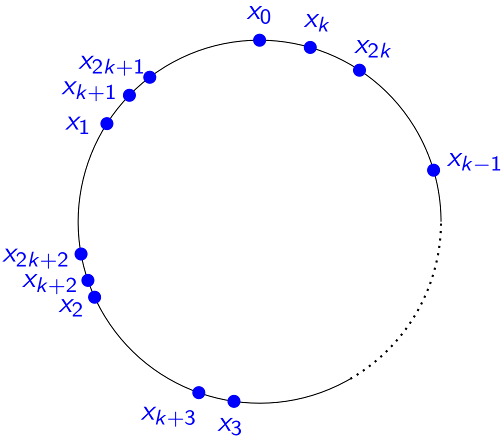

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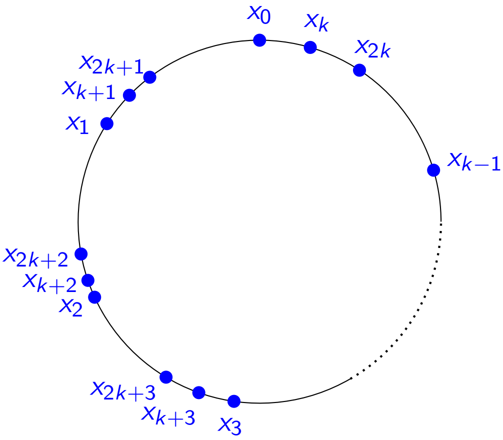

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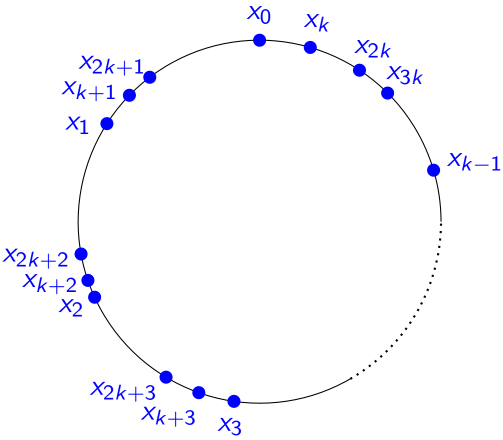

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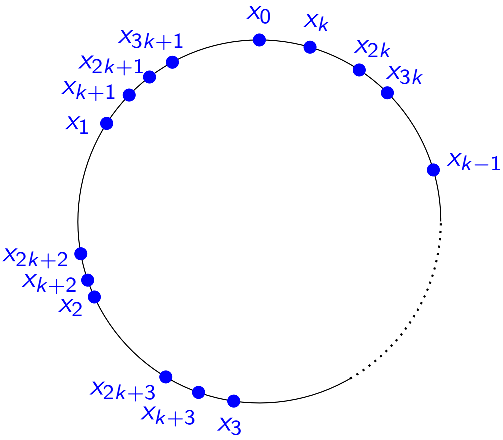

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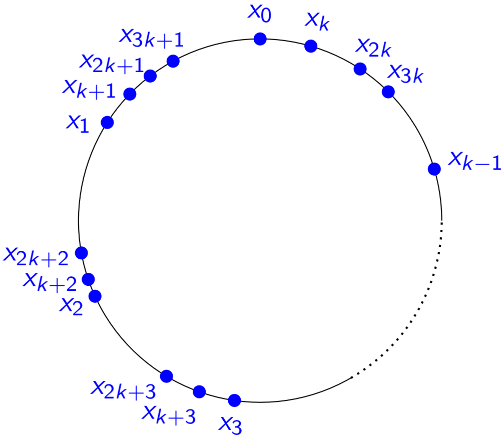

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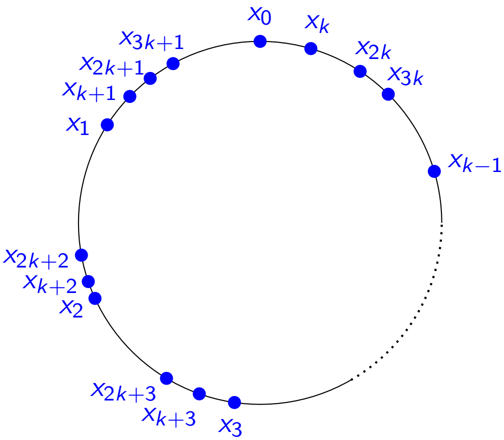

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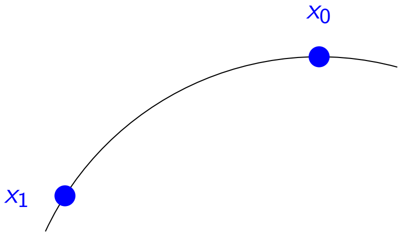

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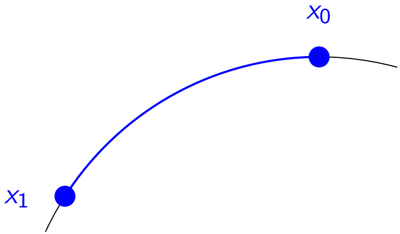

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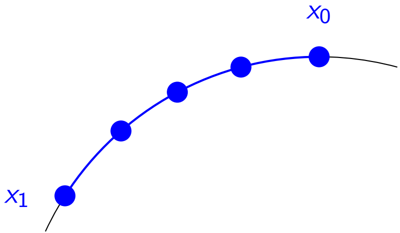

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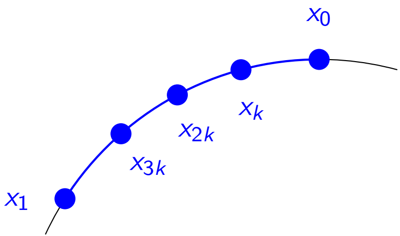

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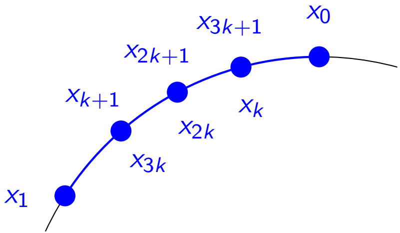

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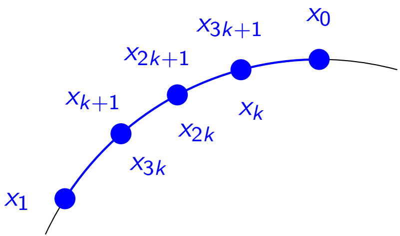

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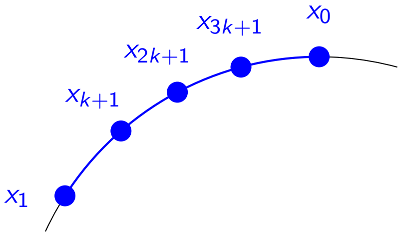

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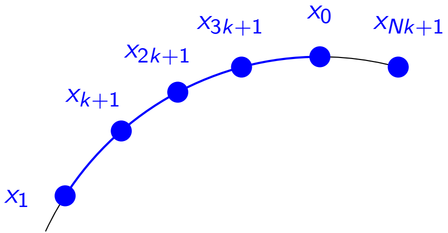

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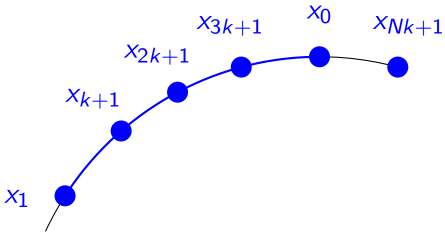

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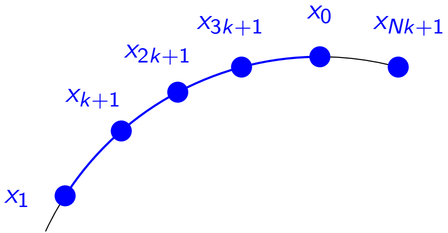

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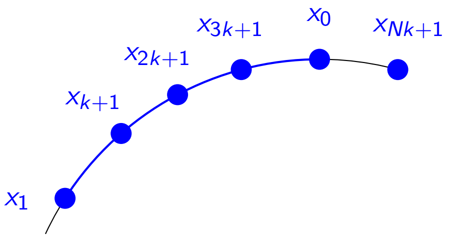
Guaranteed in $\mathcal{O}(n^3\mathbf{OPT})$ revolutions.

$\mathcal{O}(n^5\mathbf{OPT}\log n)$ runtime.



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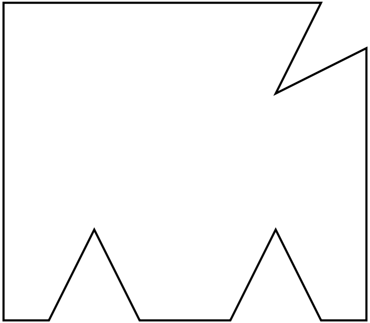

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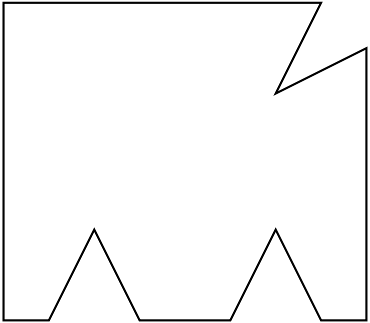

Shows existance of optimal solutions, with at least one guard at an incidence point.

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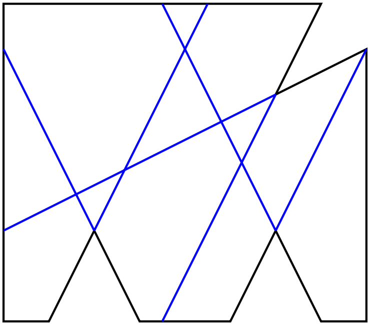

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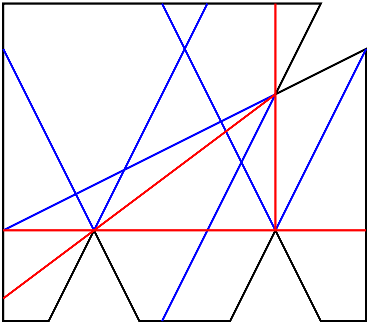

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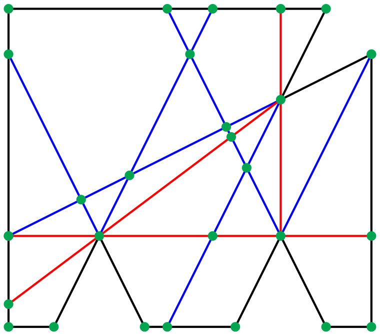

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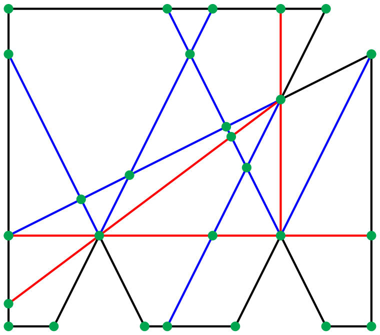

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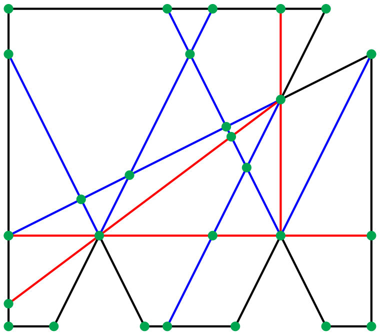

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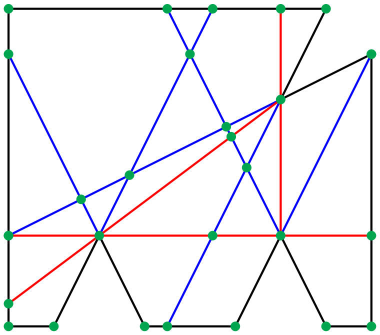

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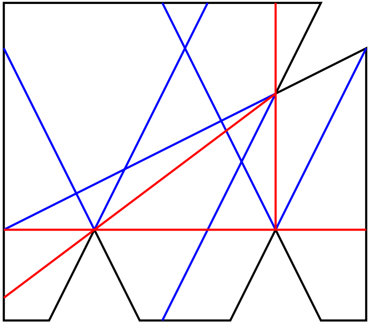
At most $\mathcal{O}(n^3)$ incidence points. Try them all!

Total running time: $\mathcal{O}(n^6 \log(n))$.

Sketch of Proof



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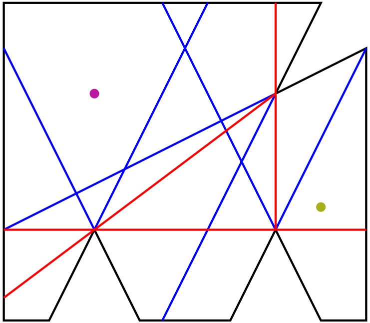

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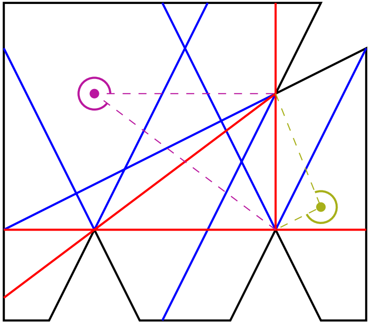

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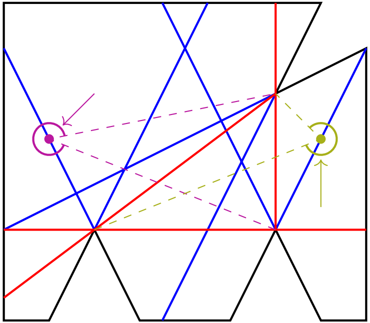

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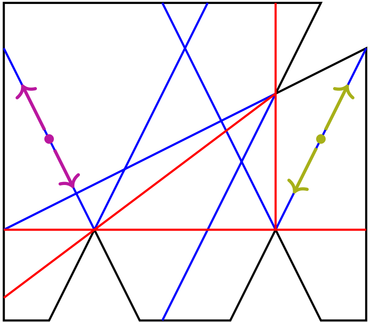

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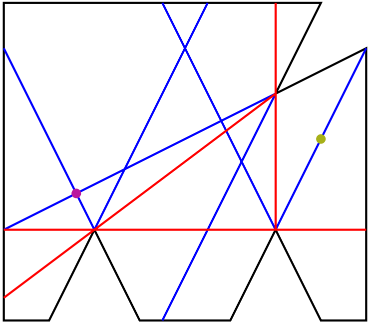

Method 3: Closed-Form Analysis ("Analytic Arc Cover Framework")

Idea: Find a closed form expression of greedy step G to evaluate *every* starting point simultaneously.

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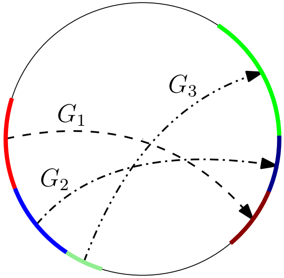


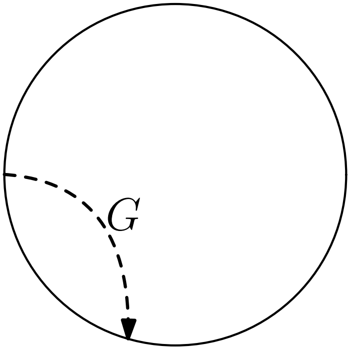
$$G_1 = \frac{ax+b}{cx+d},\ G_2 = \frac{a'x+b'}{c'x+d'},\ G_3 = \frac{a''x+b''}{c''x+d''}, \ldots$$

Bounds: $[a_0 + b_0\sqrt{c_0},\ a_1 + b_1\sqrt{c_1}),\ [a_1 + b_1\sqrt{c_1}, a_2 + b_2\sqrt{c_2}),\ [a_2 + b_2\sqrt{c_2}, a_3 + b_3\sqrt{c_3}), \ldots$

Using a closed form representation ("Analytic Arc Cover Framework")

Input: Closed form expression for G

Algorithm: Repeatedly compose G with itself...



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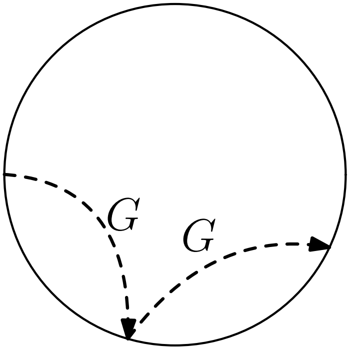

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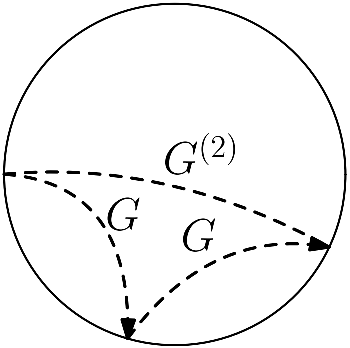

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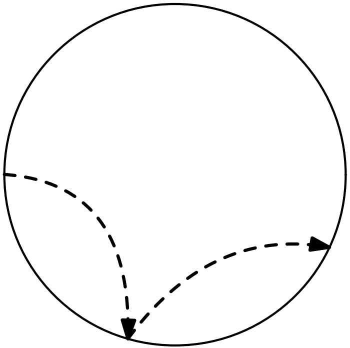

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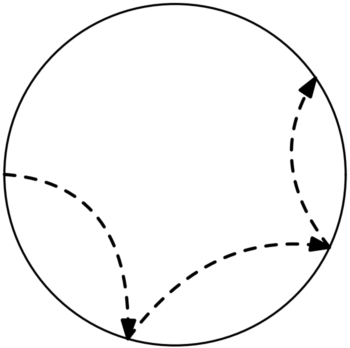

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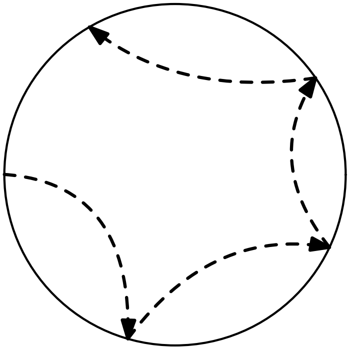

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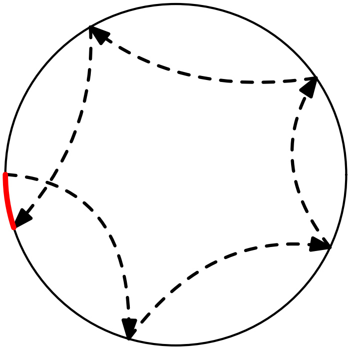
until $G^{(k)}$ exceeds a full loop somewhere.

Output: k



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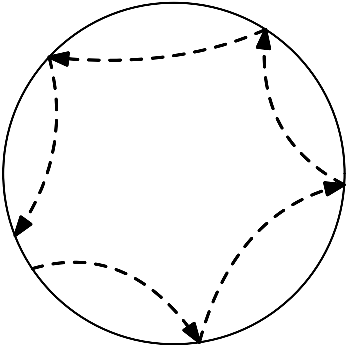

Need to test $G^{(k)}$ everywhere at once.

Analytic Arc Cover Framework: Benefits

- Only need to find one function

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- Robust to some problem variants
- Also applicable to some other problems. . .

More problems: Line Segment/Polygon Separation

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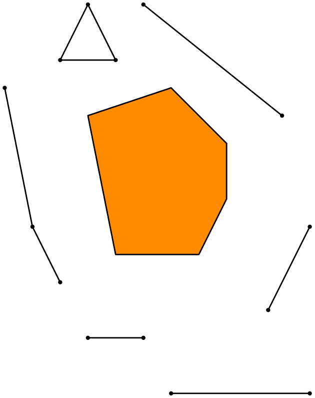

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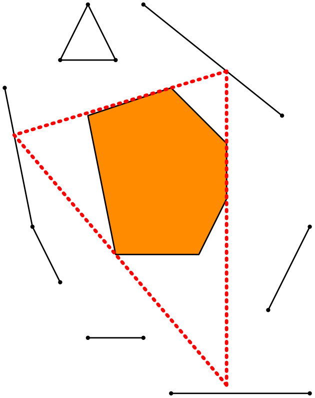

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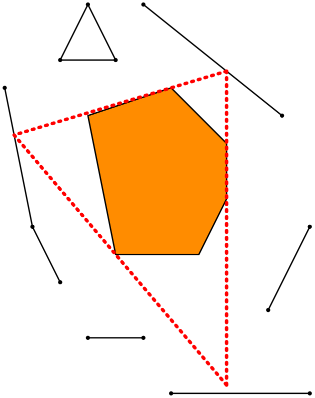

Solvable with analytic arc cover framework!

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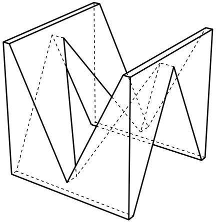
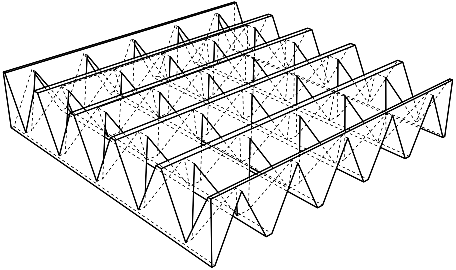
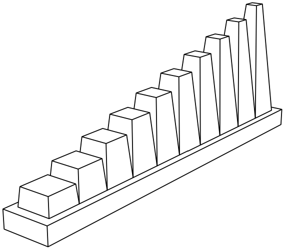


Carveable:





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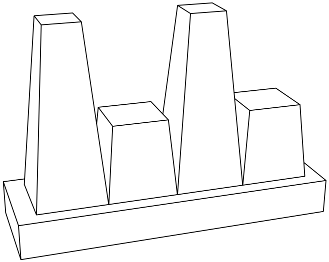
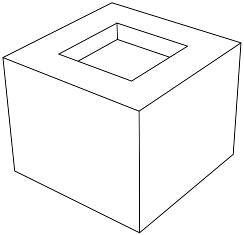
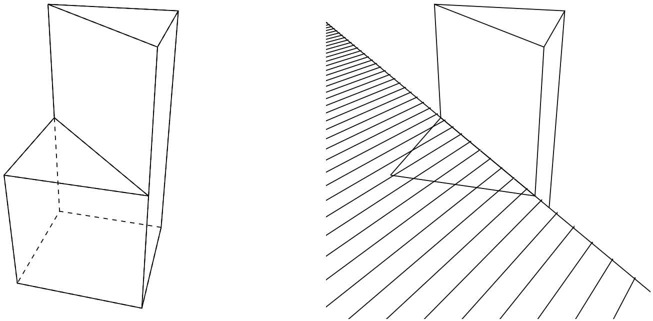
Uncarveable:





Min Half-Plane Cutting: Reducing to 2D



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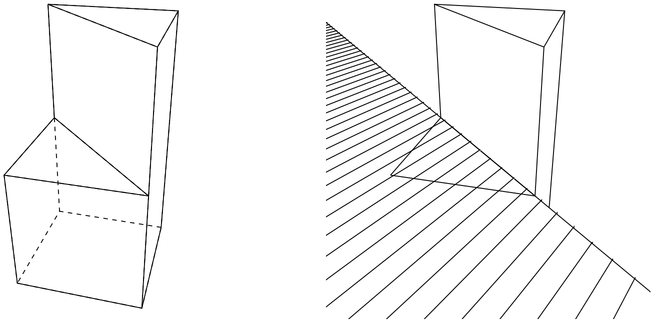

- Minimizing in 3D $\rightarrow$ Minimizing (many) in 2D
- Reduction to line segment/polygon separation

Future Work

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Faster runtime:

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- Almost linear runtime for greedy revolution

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- Way fewer revolutions for Method 1

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Other variants:

- Higher dimensional galleries.

Future Work

Faster runtime:

- Almost linear runtime for greedy revolution
- Way fewer revolutions for Method 1

Other variants:

- Higher dimensional galleries.
- Each guard guards at most m polygonal arcs.

Thank you [1]

[1] Also Jack is looking for PhD positions. . .